

An Efficient Boubakar-Chebyshev Polynomial Algorithm for High-Order Non-Linear Volterra-Fredholm Integro-Differential Equations

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ABSTRACT This study addresses the challenge of solving high-order nonlinear Volterra-Fredholm integro-differential equations (VFIDEs), which are essential in modeling complex physical and engineering systems but often pose significant computational difficulties due to their nonlinearity and mixed boundary conditions. While existing numerical methods struggle with accuracy and efficiency, this research bridges the gap by introducing a novel Boubaker-Chebyshev polynomial algorithm that leverages Chebyshev-Gauss-Radau collocation points for enhanced precision. The aim is to develop a highly accurate and computationally efficient method for solving high-order VFIDEs, addressing the shortcomings of traditional approaches. Implemented in Maple 18, the proposed algorithm is rigorously tested, demonstrating superior error estimation and computational optimization compared to conventional polynomial methods, particularly for problems with mixed boundary conditions. The results reveal marked improvements in both solution accuracy and computational performance, highlighting the effectiveness of combining Boubaker and Chebyshev polynomials. These findings have significant implications for scientific computing, offering a robust tool for tackling complex integro-differential problems. For future investigations, it is suggested to expand this method to fractional-order equations, fine-tune the selection of collocation points, and utilize it in practical nonlinear systems to further support and improve its usability. This research not only propels numerical methods for VFIDEs forward but also introduces new possibilities for high-precision computational techniques in the fields of applied mathematics and engineering.

KEYWORDS: Approximate solution, Chebyshev Gauss-Radau, Collocation points, Third Kind Chebyshev polynomial, Volterra and Fredholm integro-differential equations

I. INTRODUCTION

Integro-differential equations have become increasingly significant in modeling real-world problems across various fields such as engineering, physics, biology, and economics. The origin of the study of integral and integro differential equations may be traced to the work of Abel, Lotkab, Fredholm, Malthus, Verhust and Voltera on problems in mechanics, mathematical biology and economics (Lakshmikantham and Ramama 1995). These equations, especially the Volterra-Fredholm type, have been widely used in describing systems where the future state of the system is influenced by both its current and past states. Consequently, the study and solution of these advanced integro-differential equations have attracted substantial attention in computational mathematics (Berenguer, Gamez, La Torre, and Galan 2014). In recent years, numerous researchers have applied various numerical methods for solving differential and integral equations, especially Volterra and Fredholm types. Tinggang and Yongjun (2016) used Boubaker polynomials to solve numerical differential equations, introducing a Boubaker-Legendre transform to connect Boubaker and shifted Legendre polynomial expansions. This approach exhibited efficacy in addressing boundary value problems, notwithstanding the necessity for additional validation, particularly in error estimation. Adesanya et al. (2020) applied Boubaker polynomials in a collocation method to solve linear Fredholm-Volterra integral equations, thereby transforming them into a system of linear equations with established precision. Similarly, for first-order linear integral equations, Bushra et al. (2020) applied a Boubaker-based collocation method, proving its effectiveness with illustrative examples. While Aduroja et al. (2022) applied a Caputo-based collocation method for multi-order fractional differential equations, Ahmed et al. (2019) stressed

the superiority of the Modified Adomian Decomposition Method (MADM) over the Adomian Decomposition Method (ADM) for mixed Volterra-Fredholm equations and Anakira et al. (2023) combined MADM, ADM, Laplace transforms, and Pade approximants, demonstrating reliable solutions for a wide range of linear and non-linear Volterra equations.

Numerous scholarly investigations have concentrated on higher-order equations, including the work of Salah and Saied (2012), who employed the differential transformation method, and Babaaghaie and Maleknejad (2017), who utilized the Galerkin method, both demonstrating remarkable precision. Kareem (2011) and Agidi and Kumar (2010) further developed the Boubaker Polynomial Expansion Scheme for equations such as the neutron transport equation and Love's integral equation, while more contemporary research conducted by Sara and Jalil (2017) and Salih and Sunil (2021) broadened the application of Boubaker polynomials to numerical techniques in Laplace transforms and orthogonalization processes. Chebyshev polynomials have gained significant attention, with Gamze et al. (2012) proposing a Chebyshev collocation methodology for high-order Fredholm-Volterra equations, resulting in effective solutions devoid of intricate algorithms. Zineb et al. (2023) further enhanced this approach by devising a spectral collocation technique for fractional integro-differential equations utilizing Chebyshev-Gauss quadrature. Ayinde et al. (2021, 2022) investigated third-kind Chebyshev polynomials, which exhibited superior accuracy compared to Laguerre polynomials in the context of Volterra integro-differential equations. Additional contributions include Mustafa et al. (2012), who implemented power series for integro-differential equations, and Etuk et al. (2021), who introduced monomonic Chebyshev polynomials of the second kind for mixed Volterra-Fredholm equations. Ogunrinde et al. (2023) formulated a linear multistep method utilizing Newton-Cotes quadrature for Fredholm equations, thereby validating its stability and convergence properties. Lastly, Oyedepo et al. (2024) employed Vieta-Lucas polynomials in the analysis of Volterra equations, showcasing remarkable precision with this innovative technique. This research underscores the increasing sophistication and variety of polynomial-based numerical methods, confirming their effectiveness and adaptability in tackling diverse integral and integro-differential equations. Although other methods mostly combine these strategies with conventional collocation techniques, this study suggests a unique strategy that integrates Boubaker and third-kind Chebyshev polynomials as basis functions and uses Chebyshev Gauss-Radau collocation sites. The outstanding work of these researchers inspired us to adopt numerical Chebyshev Gauss-Radau collocation points for solving high-order nonlinear Volterra-Fredholm integro-differential equations using Boubaker and third-kind Chebyshev polynomials.

II. LITERATURE REVIEW

A. INTEGRO-DIFFERENTIAL EQUATION

This is a mathematical equation which contains derivatives of the required function and its integral transform (Zaltsman, 2011). According to Wazwaz 2011, it can also be defined as an equation in which the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. There are two kinds of integro-differential equations and they are Volterra and Fredholm integro differential equations. The major difference between the two kinds is that Volterra integro differential equation upper bound of integration is a variable while that of Fredholm have a constant upper bound. The general form of the Volterra integro-differential equation considered in this study is expressed as:

$$\omega^n(t) = f(t) + \gamma \int_a^t K(x, t) \omega(x) dx \quad (1)$$

and the general form of the Fredholm integro-differential equation is:

$$\omega^n(t) = f(t) + \gamma \int_a^b K(x, t) \omega(x) dx \quad (2)$$

where $\omega^n(t) = \frac{d^n \omega}{dt^n}$ represents the n^{th} derivative of the unknown function $f(x)$ is a given function, λ represents a constant parameter, $K(x, t)$ is the kernel of the integral term, $\omega(x)$ represents the unknown function evaluated at a variable t and a and b are constants of integration.

B. COLLOCATION

These are ways of approximately solving differential equations (James, 2022). It is a method of evaluating an approximate solution in a suitable collection of functions known as trial solution (Ayinde et al., 2022). In using the above-mentioned numerical collocation method, we employed the following Gauss Radau collocation point: Where the Chebyshev Gauss Radau points allocations are given by

$$t_i = \cos\left(\frac{2\pi i}{2m+1}\right), i \geq 0 \quad (3)$$

m is the number of nodes. m numbers of equations that will be generated.

C. BOUBAKAR POLYNOMIAL

According to (Sara & Jalil, 2017) is defined by

$$\beta_i(t) = \sum_{j=0}^{i/2} (-1)^j \binom{i-j}{j} \left(\frac{i-4j}{i-j}\right) t^{i-2j} \quad i \geq 0 \quad (4)$$

Hence;

$$\left. \begin{aligned} \beta_0(t) &= \sum_{j=0}^0 (-1)^j \binom{-j}{j} \left(\frac{-4j}{-j}\right) t^{-2j} = 1 \\ \beta_1(t) &= \sum_{j=0}^0 (-1)^j \binom{1-j}{j} \left(\frac{1-4j}{1-j}\right) t^{1-2j} = t \\ \beta_2(t) &= \sum_{j=0}^1 (-1)^j \binom{2-j}{j} \left(\frac{2-4j}{2-j}\right) t^{2-2j} = t^2 + 2 \\ \beta_3(t) &= \sum_{j=0}^1 (-1)^j \binom{3-j}{j} \left(\frac{3-4j}{3-j}\right) t^{3-2j} = t^3 + t \\ &\vdots \end{aligned} \right\} \quad (5)$$

and the recurrence formular is:

$$\beta_i(t) = t\beta_{i-1}(t) - \beta_{i-2}(t) \quad \forall i > 2 \quad (6)$$

D. THIRD-KIND CHEBYSHEV POLYNOMIALS

Third-Kind Chebyshev Polynomials $\gamma_i(t)$ according to Eslahchi et al., 2024 over the interval $[-1,1]$ is defined by

$$\gamma_i^3(t) = \frac{\cos\left(i+\frac{1}{2}\right)\delta}{\cos\left(\frac{\delta}{2}\right)} \quad i \geq 0 \quad (7)$$

where $t = \cos(\delta)$

$$\left. \begin{aligned} \gamma_0^3(t) &= 1 \\ \gamma_1^3(t) &= 2t - 1 \\ \gamma_2^3(t) &= 4t^2 - 2t - 1 \\ \gamma_3^3(t) &= 8t^3 - 4t^2 - 4t + 1 \\ &\vdots \end{aligned} \right\} \quad (8)$$

The recurrence formula is:

$$\gamma_{i+1}(t) = 2t\gamma_i(t) - \gamma_{i-1}(t), \forall i > 1 \quad (9)$$

E. EXACT SOLUTION

This is a symbolic representation of a value which solves given equation exactly (Alex, 2021). It is the solution that may be expressed in a closed form (Ayinde et al., 2022).

F. APPROXIMATE SOLUTION

This is also called the trial solution and it's denoted by $\omega_i(t)$ is given in the form:

$$\omega_m(t) = \sum_{i=0}^m \alpha_i \beta_i(t) \quad (10)$$

where $\alpha_i (i \geq 0)$ are to be determined.

G. APPROXIMATE SOLUTION

This is the process of estimating the absolute error that is, the absolute difference between the exact and the approximate solution. To estimate error the approximate solution $\hat{\omega}_i$ will be compared with the actual solution ω_i to obtain the absolute errors e_i :

$$e_i = |\omega_i - \hat{\omega}_i| \quad (11)$$

III. METHODOLOGY

In this section, we considered the general form of the Volterra-Fredholm integro-differential equation of the form:

$$\alpha_{01}\omega^m(t) + \alpha_{11}\omega^{m-1}(t) + \alpha_{21}\omega^{m-2}(t) \dots + \alpha_{m1}\omega(t) + \gamma \int_{p(t)}^{q(t)} K(x,t)\omega^m(x)dx = g(t) \quad (12)$$

where $\omega^m(t) = \frac{d^m \omega}{dt^m}$ are the derivative and there exist an initial condition $\omega(0), \omega'(0), \omega''(0), \dots, \omega^{n-1}(0)$ to determine the particular solution of the equation.

A. CHEBYSHEV GAUSS-RADAU COLLOCATION POINT BY BOUBAKAR POLYNOMIALS

Here, we assume a trial solution of the form:

$$\omega_m(t) = \sum_{i=0}^m \alpha_i \beta_i(t) \quad (13)$$

where m is the degree of our approximant, $\alpha_i, i = 0, 1, 2, \dots, m$ are unknown constants to be determined and $\beta_i(t)$ ($i \geq 0$) are Boubaker polynomials, which serve as the approximating polynomials defined earlier in the study to solve the high-order problem given in equation (12), along with the initial conditions specified.

Thus, differentiating equation (13) n times, we obtain:

$$\left. \begin{aligned} \omega'_m(t) &= \sum_{i=0}^m \alpha_i \beta'_i(t), \omega''_m(t) = \sum_{i=0}^m \alpha_i \beta''_i(t), \omega'''_m(t) = \sum_{i=0}^m \alpha_i \beta'''_i(t), \dots, \\ \omega_m^{(n)}(t) &= \sum_{i=0}^m \alpha_i \beta_i^{(n)}(t) \end{aligned} \right\} \quad (14)$$

Substituting equations (13) and (14) into equation (1), we obtain:

$$\alpha_{01} \sum_{i=0}^m \alpha_i \beta_i^m(t) + \alpha_{11} \sum_{i=0}^m \alpha_i \beta_i^{m-1}(t) + \alpha_{21} \sum_{i=0}^m \alpha_i \beta_i^{m-2}(t) + \dots + \alpha_{m1} \sum_{i=0}^m \alpha_i \beta_i(t) + \gamma \int_{p(t)}^{q(t)} K(x,t) \sum_{i=0}^m \alpha_i \beta_i^m(x) dx = g(t) \quad (15)$$

Evaluating the integral part of equation (15) and simplifying, we get:

$$\alpha_{01} \sum_{i=0}^m \alpha_i \beta_i^m(t) + \alpha_{11} \sum_{i=0}^m \alpha_i \beta_i^{m-1}(t) + \alpha_{21} \sum_{i=0}^m \alpha_i \beta_i^{m-2}(t) + \dots + \alpha_{m1} \sum_{i=0}^m \alpha_i \beta_i(t) + \gamma G(t) = g(t) \quad (16)$$

where:

$$G(t) = \int_{p(t)}^{q(t)} K(x,t) \sum_{i=0}^m \alpha_i \beta_i^m(x) dx \quad (17)$$

Thus, collocating equation (16) at points $t = t_i$, we obtain

$$\alpha_{01} \sum_{i=0}^m \alpha_i \beta_i^m(t_i) + \alpha_{11} \sum_{i=0}^m \alpha_i \beta_i^{m-1}(t_i) + \alpha_{21} \sum_{i=0}^m \alpha_i \beta_i^{m-2}(t_i) + \dots + \alpha_{m1} \sum_{i=0}^m \alpha_i \beta_i(t_i) + \gamma G(t_i) = g(t_i) \quad (18)$$

Where:

$$t_i = \cos\left(\frac{2\pi i}{2m+1}\right) \quad i = 0, 1, 2, \dots, m \quad (19)$$

This equation can then be expressed in matrix form as:

$$\phi \alpha = g \quad (20)$$

$$\text{where: } \phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & \dots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{11}^0 & \phi_{12}^0 & \phi_{13}^0 & \phi_{14}^0 & \dots & \phi_{1n}^0 \\ \phi_{21}^0 & \phi_{22}^0 & \phi_{23}^0 & \phi_{24}^0 & \dots & \phi_{2n}^0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}^{n-1} & \phi_{m2}^{n-1} & \phi_{m3}^{n-1} & \phi_{m4}^{n-1} & \dots & \phi_{mn}^{n-1} \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}, g = \begin{bmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{mn}^{n-1} \end{bmatrix}$$

Here, ϕ_i 's represent the coefficients of α_i 's, and $g(t_i)$ are the given functions. The matrix inversion approach is then used to solve the system of equations to obtain the unknown constants.

$$\alpha = \phi^{-1}g \quad (21)$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \dots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} & \dots & \phi_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{11}^0 & \phi_{12}^0 & \phi_{13}^0 & \phi_{14}^0 & \dots & \phi_{1n}^0 \\ \phi_{21}^0 & \phi_{22}^0 & \phi_{23}^0 & \phi_{24}^0 & \dots & \phi_{2n}^0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}^{n-1} & \phi_{m2}^{n-1} & \phi_{m3}^{n-1} & \phi_{m4}^{n-1} & \dots & \phi_{mn}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{mn}^{n-1} \end{bmatrix}$$

Equation (18) gives rise to $m+1$ algebraic linear equations with $m+1$ unknown constants. Two additional equations are obtained from the conditions specified. Altogether, we now have $m+2$ unknown constants. Using Maple 18, these equations are solved to determine the $m+2$ unknown α_i constants ($i \geq 0$), which are then substituted back into the approximate solution given by equation (13).

B. CHEBYSHEV GAUSS-RADAU COLLOCATION POINT BY THIRD KIND CHEBYSHEV POLYNOMIALS

To explore this approach, we assume an approximate solution of the form:

$$\omega_m(t) = \sum_{i=0}^m \alpha_i \gamma_i(t), \quad i \geq 0 \quad (22)$$

where α_i are unknown constants to be determined, m is the degree of approximation, and $\gamma_i(t)$ represents the approximating polynomials of the Third-Kind Chebyshev Polynomials. This approach is similar to the previously described in section 3.2. We obtain an algebraic linear system of equations with $m+1$ equations. The $m+1$ algebraic linear system of equations is then solved using MAPLE 18 software to determine the unknown constants α_i ($i \geq 0$) which are subsequently substituted back into the approximate solution given by equation (13).

C. NUMERICAL EXAMPLES

In this section, we have demonstrated the Chebyshev Gauss-Radau collocation point on the nonlinear high-order integro-differential equations using Boubakar and Third kind Chebyshev polynomial as the basis function. The results obtained are compared with each other on three examples to test for the effectiveness and efficiency of our collocation points via the Maple 18 software.

NUMERICAL EXAMPLE 1

Consider the third-order nonlinear integro-differential equation of the form (Salah, 2012)

$$\omega^{iii}(t) + \omega(t) = -\frac{t^5}{5} + \frac{2t^3}{3} + \frac{5t^2}{6} - \frac{113t}{105} - 1 + \int_0^t \omega^2(x)dx + \int_0^1 xt(x+t) \omega^2(x)dx \quad (23)$$

Subject to the initial conditions

$$\omega(0) = -1 \quad \omega^i(0) = 0 \quad \text{and} \quad \omega^{ii}(0) = 2 \quad (24)$$

Exact solution is

$$\omega(t) = -1 + t^2 \quad (25)$$

NUMERICAL EXAMPLE 2

Consider the fourth-order nonlinear integro-differential equation of the form (Salah, 2012)

$$t^4 \omega^{iiii}(t) - \omega^i(t) = \frac{t^6}{30} \frac{t^4}{6} \frac{t^2}{2} + \frac{14t}{3} - \frac{1}{2} \int_0^t (t-x) \omega^2(x) dx - 2 \int_0^1 (x+t) \omega(x) dx \quad (26)$$

Subject to the initial conditions

$$\omega(0) = 1 \quad \omega^i(0) = 0 \quad \omega^{ii}(0) = 2 \quad \text{and} \quad \omega^{iii}(0) = 0 \quad (27)$$

Exact solution is

$$\omega(t) = 1 + t^2 \quad (28)$$

NUMERICAL EXAMPLE 3

Consider the fifth-order nonlinear integro-differential equation of the form (Salah, 2012)

$$t^3 \omega^v(t) - 2\omega^i(t) + t\omega(t) = t^4 + 5t^3 - 4e^t - \frac{(e^2-1)t}{4} + 2 + \int_0^t x\omega(x) dx + \int_0^1 t\omega^2(x) dx \quad (29)$$

Subject to the initial conditions

$$\omega(0) = 0, \quad \omega^i(0) = 1, \quad \omega^{ii}(0) = 2, \quad \omega^{iii}(0) = 3, \quad \text{and} \quad \omega^{iv}(0) = 4 \quad (30)$$

Exact solution is

$$\omega(t) = te^t \quad (31)$$

IV. RESULTS AND DISCUSSION

In this section, the utilization of Chebyshev-Gauss-Radau collocation points has been implemented to address representative examples. The numerical results derived through the current methodology have been juxtaposed with the exact solutions of the representative examples. Similarly, absolute errors of results from this present method have been compared utilizing Boubaker and Third kind Chebyshev polynomials as basis function. The following acronyms are adopted in the tables;

BP: Boubaker Polynomial

TKCP: Third Kind Chebyshev Polynomials

$$\text{Error: } E_N = |\omega(t) - \omega_N(t)| \quad (32)$$

Table 1: Exact, Boubaker polynomial (BP), Third kind Chebyshev polynomial (TKCP) and Error Estimates example 1 for Case M = 4

t	$\omega(t) = t^2 - 1$	ω_{BP} (Approx.)	ω_{TKCP} (Approx.)	E_{BP}	E_{TKCP}
0.0	-1.000000000	-1.0000000000	-1.000000000	0.0000000000	0.0000000000
0.2	-0.960000000	-0.9603634283	-0.9635966273	3.6343×10^{-4}	3.5966×10^{-3}
0.4	-0.840000000	-0.8436698281	-0.86767298419	3.6698×10^{-3}	2.7673×10^{-2}
0.6	-0.640000000	-0.6549587804	-0.72968370468	1.4959×10^{-2}	8.9768×10^{-2}
0.8	-0.36000.3600	-0.4015570678	-0.56378331895	4.1557×10^{-2}	2.0380×10^{-1}
1.0	0.000000000	-0.0930786831	-0.38082625313	9.3079×10^{-2}	3.8080×10^{-1}

Table 2: Exact, Boubaker polynomial (BP), Third kind Chebyshev polynomial (TKCP) and Error Estimates example 2 for Case M = 5

t	$\omega(t) = t^2 + 1$	ω_{BP} (Approx.)	ω_{TKCP} (Approx.)	E_{BP}	E_{TKCP}
0.0	1.0000000000	1.0000000000	1.0000000010000	0.0000000000	1.1000×10^{-9}
0.2	1.0400000000	1.0400000001	1.0400000010158	0.0000000000	1.1000×10^{-9}
0.4	1.1600000000	1.1600000002	1.1600000009753	0.0000000000	1.1000×10^{-9}
0.6	1.3600000000	1.3600000004	1.3600000007434	0.0000000000	1.1000×10^{-9}
0.8	1.6400000000	1.6400000007	1.6400000001144	1.1000×10^{-9}	0.0000000000
1.0	2.0000000000	2.0000000013	1.999999998281	1.1000×10^{-9}	1.1000×10^{-9}

Table 3: Exact, Boubaker Polynomial (BP), Third kind Chebyshev polynomial (TKCP) approximate solution and Error Estimates for example 2 for Case M = 5

t	$\omega(t) = te^t$	ω_{BP} (Approx.)	ω_{TKCP} (Approx.)	E_{BP}	E_{TKCP}
0.0	0.0000000000	0.25462668145	4.8871×10^{-10}	2.4000×10^{-11}	4.8871×10^{-10}
0.2	0.244280551632	0.61773078411	0.244269519901	1.0346×10^{-2}	1.1032×10^{-5}
0.4	0.596729879057	1.12338926592	0.596361979547	2.1001×10^{-2}	3.6781×10^{-4}
0.6	1.093271280234	1.81162078047	1.090354337334	3.0118×10^{-2}	2.9169×10^{-3}
0.8	1.780432742794	1.64000000007	1.767574208856	3.1188×10^{-2}	1.2859×10^{-2}
1.0	2.718281828459	2.72828479412	2.677153465521	1.0021×10^{-2}	4.1128×10^{-2}

The results from Table 1 ($M = 4$) demonstrate that the Boubaker Polynomial (BP) outperforms the Third Kind Chebyshev Polynomial (TKCP) in approximating the exact solution, exhibiting significantly smaller error estimates and faster convergence. This suggests that BP's orthogonality properties, boundary alignment, or spectral efficiency make it particularly well-suited for this problem at lower polynomial degrees. The trend continues in Table 2 ($M = 5$), where BP maintains its superior accuracy, reinforcing its reliability for this class of problems. However, Table 3 ($M = 5$, Example 2) reveals a notable shift: TKCP now delivers smaller errors and better convergence than BP, indicating that its adaptive weighting or orthogonal structure may offer advantages under specific conditions. Collectively, these findings highlight that while BP generally performs better in earlier cases, TKCP can surpass it depending on problem-specific factors such as boundary conditions or solution smoothness. Further investigation is needed to determine the optimal polynomial choice for varying parameters and higher-order approximations.

V. CONCLUSION

The error analysis, as presented in table 1 to 3, indicate that, for lower degrees of approximation specifically when $m = 4$ and $m = 5$, the Boubaker polynomial approach yielded slightly lower errors than the third-kind Chebyshev polynomial. This suggests that the Boubaker polynomial may offer better initial accuracy for certain VFIDEs at these approximation levels. However, as the degree of approximation increased, the Boubaker polynomial outperformed the third-kind Chebyshev polynomial, achieving a lower error margin and thus demonstrating superior accuracy in higher-order approximations. In conclusion, this research provides valuable insights into the selection of polynomial methods for solving VFIDEs. The third-kind Chebyshev polynomial is recommended for higher-order approximations, while the Boubaker polynomial may be more suitable for lower-order cases.

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REFERENCES

- Abdul-Majid, W. (2011). *Linear and Nonlinear Integral Equations, Method and Applications*. Higher Education Press, Beijing. ISBN 978-7-04-031694-0.
- Abubakar, A., & Taiwo, O. A. (2014). Integral Collocation Approximation Methods for the Numerical Solution of High Order Linear Fredholm Volterra Integro Differential Equations. *American Journals of Computational and Applied Mathematics*, 4(4), 111-117. <https://doi.org/10.5923/j.ajcam.20140404.01>
- Adesanya, A. O., Yahaya, Y. A., Ahmed, B., & Onsachi, R. O. (2020). Numerical solution of linear integer and integro differential equations using Boubakar collocation method. *International Journal of Mathematical Sciences and Optimization: Theory and Application*, 2019(2), 592-598.
- Aduroja, O. O., Agbolade, O., & Adiku, L. (2022). Numerical method for the solution of multi-order fractional differential equations. *FUOYE Journal of Management Innovation and Entrepreneurship*. 1(2), 194-198
- Agidi, M., Kumar, A. S., & Boubaker, A. (2010). Polynomial Expansion Scheme Solution to Random Love Equation in the case of a rational kernel. *Electronic Journal of Theoretical Physics*, 24, 319-326.
- Ahmed, A. H., Nedal, M. M., Kirtiwan, P. G., & Satyawan, L. D. (2019). Solving integro differential equations by using numerical techniques. *International Journal of Applied Engineering Research*, 14(14), 3219-3225.
- Al-Humedi, H. O., & Shousan, A. F. (2021). A combination of the Orthogonal Polynomial with Least Squares Method for Solving High Order Fredholm Volterra Integro Differential Equations. *AlQadisyah Journal of Pure Science*, 26(1), 20-38.
- Ami, M. S., Mardy, & Emad, M. H. (2017). On the Numerical Solution of Hammerstein integral Equations using Shifted Chebyshev Polynomials of the third kind method. *New Trend in Mathematical Sciences*, 5, 273-283.
- Anakira, N. R., Adel, A., Katabeh, D., Hani, G. B., Jameel, A. F., Khamis, S. A., Kalbani, & Abu Danas, M. (2023). An Algorithm for solving Linear and Non Linear Volterra- Integro differential equations. *International Journal Advance Soft Computing Applications*, 15(3), 2074-8523
- Asiya, A., & Najmuddin, A. (2023). Numerical Solution for nonlinear Volterra-Fredholm Integro-differential Equations using Adomian and Modified Adomian Decomposition Method. *Research Square*, 31(2) 1-15
- Ayinde, A. M., James, A. A., Ishaq, A. A., & Oyedapo, T. (2022). A new numerical approach using Chebyshev Third Kind Polynomial for Solving Integro Differential Equations of Higher Order. *GUJ Science, Part A*, 9(3), 259-266.
- Ayinde, A. M., Jimoh, A. K., Momoh, A. A., & Aliyu, H. B. (2021). Numerical Approximation Method for Solving Integro Differential Equations Via Third Kind Chebyshev and Laguerre Polynomials. *Bima Journal of Science & Technology*, 5(1), 192-201
- Babaaghaie, A., & Maleknejad, K. (2019). Numerical solution of integro differential equations of high order by wavelet basis, its algorithm and convergence analysis. *Journal of Computational and Applied Mathematics*, 325, 125-133. <https://doi.org/10.1016/j.cam.2017.04.035>.
- Berenguer, M. I., Gamez, D., Kunze, H., La Torre, D., & Galán, M. R. (2014). Solving direct and inverse problems for Fredholm-type integro-differential equations with application to pollution diffusion modeling. *Mathematics and Computers in Simulation*, 223, 394-404.
- Bhrawu, A. H., Abdelkawu, M. A., Alzahrani, A. A., Baleanu, D., & Alzahrani, E. O. (2015). Chebyshev Laguerre Gauss Radau Collocation Scheme for solving a time fractional sub-diffusion equation on a semi-infinite domain. *Proceedings of the Romanian Academy, Series A*, 16(4), 490-498.
- Boubaker, K. (2011). Boubakar Polynomial Expansion Scheme (BPES) Solution to Boltzmann diffusion equation the case of strongly anisotropic neutral particles forward-backward scattering. *Annals of Nuclear Energy*, 38(8), 1715-1717. <https://doi.org/10.1016/j.anucene.2011.04.009>.
- Bushra, E. K., Eman, H. O., Semaa, H. A., Muhammed, R., & Suha, S. (2020). Some results for Orthonormal Boubakar polynomial with application. *Journal of Southwest Jiaotong University*, 55(3), 26. <https://doi.org/10.35741/issn.0258-2724.55.3.26>
- El-Sayed, A. M., Gadelhak, M., & Ragab, A. A. (2018). On the Numerical Solutions of High-Order Volterra-Fredholm Integro-Differential Equations. *Applied Mathematics and Computation*, 327, 314-326.
- Eslahchi, M. R., Mehdi, D., & Sanaz, A. (2012). The third and fourth kinds of Chebyshev Polynomials and best Uniform Approximation. *Mathematical and Computer Modelling*, 55(5-6), 1746-1762.