

# A Class of Continuous 3-Step Method For Solving General Second Order Differential Equations

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## Abstract

This research paper introduces a Class of Continuous 3-Step Method for Solving Second (CCSMSS) Order Ordinary Differential Equations. The approach utilizes power series as the basis function to interpolate at grid and off-grid points, while collocating the associated differential system at all grid points to resolve the problem. The primary predictors necessary for evaluating implicit methods are determined to be of the same order as the method at any hybrid collocation point. To demonstrate the method's efficacy and practicality, its characteristics are examined for consistency, stability and convergence and subsequently applied to both linear and nonlinear problems, yielding results that surpass the accuracy of existing techniques.

**Keywords:** Linear multistep, interpolation techniques, predictor-corrector, Power series.

**Mathematics Subject Classification:** 65D05

## 1 Introduction

This study investigates the numerical solution of a second-order initial value problem expressed as:

$$y'' = f(x, y, y'); y(a) = y_0; y'(a) = y_1 \quad (1)$$

It is commonly understood that such equations can be resolved through direct or indirect methods, or by transforming the problem into a set of first-order differential equations before applying alternative approaches. Equation (1) emerges from various physical processes across numerous applications, particularly in engineering fields such as vehicle motion (rockets or satellites), electrical circuits, and fluid dynamics, among others. Gholamtabar Lambert (1973), Kayode and Adegboro (2018), Obarhua and Adegboro (2021), Chan et al. (2014). Additional techniques have been developed using exponential fitting methods (refer to Simos (1998a, 2002), Van de Vyver (2005a), Van de Vyver (2006b), Monovasilis et al. (2013), and Nguyen et al. 2007, Adegboro (2022). Exponentially-fitted methods are believed to outperform polynomial-based approaches when the frequency or a reliable estimate is known in advance. Though Lambert (1973) solved second order by reducing it to system of first order differential equations (ODEs) but (CCSMSS) was able to solve directly without reducing to system of first (ODEs). Numerical experiments demonstrate the efficacy of 3-Step solutions for solving ordinary differential equations numerically. Conversely, the (CCSMSS) introduced in this study eliminates the need for calculating higher-order derivatives, which can increase computational costs, burden on computer time particularly in nonlinear systems. This paper proposes a third-order (CCSMSS) and extends its application to both linear and nonlinear problems. The subsequent paragraphs outline the article's structure. Section 2 focuses on developing the (CCSMSS) to address the issue. Section 3 covers the analysis and implementation of the (CCSMSS). Numerical examples in Section 4 showcase the accuracy and efficiency of the CCSMSS, followed by results and discussion in Section 5. The paper concludes with Section 6.

## 2 Derivation of the Method

The Continuous Collocation Spline Multistep Scheme (CCSMSS) is developed by approximating the exact solution  $y(x)$  with an estimated solution  $y(x, u)$ , which inherently produces a discrete method. The process employs a basis function in the form:

$$p(x) = \sum_{j=0}^{(c+i)-1} a_j x^j \quad (2)$$

This function is utilized to address second-order initial value problems. The second derivative of equation (2) is expressed as:

$$y''(x) = \sum_{j=0}^{(c+i)-1} j(j-1)a_j x^{j-2} = f(x, y, y') \quad (3)$$

By interpolating equation (2) at  $x_{n+j}, j = 0, k-1$ , and collocating equation (3) at  $x_{n+j}, j = 0(3)k$ , a system of  $k+3$  equations is generated:

$$y(x_{n+j}, u) = y_{n+j}, j = 1(\frac{1}{2}) \quad (4)$$

$$\frac{d^2}{dx^2}(y(x_{n+j}, u)) = f_{n+j}, j = 0(3)k \quad (5)$$

Equations (2) and (3) yield a set of  $3k$  system equations, which are solved using Cramer's rule to determine the  $a'_j$ s values. These values are then incorporated into equation (2) to create the continuous CCSMSS. After some algebraic manipulation, the CCSMSS is formulated. The equation  $y(x)$  is expressed as:

$$y(x) = \alpha_{n+1}(x) + \alpha_{n+\frac{1}{2}}(x) + h^2(\beta_n(x, w)f_n + \beta_{n+1}(x)f_{n+1} + \beta_{n+2}(x)f_{n+2} + \beta_{n+3}(x)f_{n+3}) \quad (6)$$

In this equation,  $\alpha_n(x), \alpha_{n+2}(x), \beta_n(x), \beta_{n+1}(x), \beta_{n+2}(x), \beta_{n+3}(x)$  represent continuous coefficients. These coefficients in Equation (6) are utilized to create the method outlined in Equation (2). By evaluating (6) at  $x = x_{n+3}$ , the coefficients are defined as follows:

$$\alpha_{\frac{1}{2}} = -2t - 2 \quad (7)$$

$$\alpha_1 = 2t + 3 \quad (8)$$

$$\beta_0 = -\frac{h^2(48t^5 - 160t^3 + 127t + 15)}{5760} \quad (9)$$

$$\beta_1 = \frac{h^2(48t^5 + 80t^4 - 320t^3 + 1537t + 1185)}{1920} \quad (10)$$

$$\beta_2 = -\frac{h^2(48t^5 + 160t^4 - 160t^3 - 960t^2 - 973t - 535)}{1920} \quad (11)$$

$$\beta_3 = \frac{h^2(48t^5 + 240t^4 + 320t^3 - 203t - 75)}{5760} \quad (12)$$

The first derivatives of equations (7-13) are as follows:

$$\alpha'_1 = 2 \quad (13)$$

$$\alpha'_{\frac{1}{2}} = -2 \quad (14)$$

$$\beta'_0 = -\frac{h^2(240t^4 - 480t^2 + 127)}{5760} \quad (15)$$

$$\beta'_1 = \frac{h^2 (240 t^4 + 320 t^3 - 960 t^2 + 1537)}{1920} \quad (16)$$

$$\beta'_2 = -\frac{h^2 (240 t^4 + 640 t^3 - 480 t^2 - 1920 t - 973)}{1920} \quad (17)$$

$$\beta'_3 = \frac{h^2 (240 t^4 + 960 t^3 + 960 t^2 - 203)}{5760} \quad (18)$$

When equations (7-18) are evaluated at  $t = 1$ , the resulting discrete method is:

$$y_{n+3} = \frac{h^2}{192} (-f_n + 253 f_{n+1} + 217 f_{n+2} + 11 f_{n+3}) + 5 y_{n+1} - 4 y_{n+\frac{1}{2}} \quad (19)$$

The corresponding first derivative is:

$$y'_{n+3} = \frac{1}{5760} (113 f_n + 3411 f_{n+1} + 7479 f_{n+2} + 1957 f_{n+3}) h^2 + 11520 y_{n+1} - 11520 y_{n+\frac{1}{2}} \quad (20)$$

The approximate solution (2) was employed to develop both predictors and correctors for the scheme. While the method for deriving the predictor from the main scheme remains consistent, the interpolation and collocation points differ. For the primary predictor, Equations (2) and (3) were collocated at specific grid locations and off-grid points, then evaluated at the endpoint.

The main predictor's procedure is as follows:

$$y_{n+3} = \frac{h^2}{24} (-7 f_n + 11 f_{n+1} + 34 f_{n+2} + 22 f_{n+\frac{1}{2}}) + 5 y_{n+1} - 4 y_{n+\frac{1}{2}} \quad (21)$$

$$y'_{n+3} = \frac{h^2}{720} (-1209 f_n - 3243 f_{n+1} + 2158 f_{n+2} + 3914 f_{n+\frac{1}{2}}) + 1440 y_{n+1} - 1440 y_{n+\frac{1}{2}} \quad (22)$$

### 3 Analysis Properties of CCSMSS

To prove the correctness of the derived approaches, their fundamental properties are examined. These characteristics namely, error constant, consistency, and zero stability, show how the approaches converge. For a better understanding of the chapter, these qualities are briefly introduced in the sections that follow.

#### Order And Error Constant Of The Methods

Let the linear operator  $L$  associated with the new method (19) be defined as

$$L[y(x); h] = \sum_{j=0}^k \alpha_j y(x_n + jh) - h^2 \beta_j y''(x_{n+jh}) \quad (23)$$

where  $y(x)$  is an arbitrary test function that is continuously differentiable in the interval  $[a, b]$ .

Expanding  $y(x_n + jh)$  and  $y''(x_n + jh)$ ,  $j = 0, 1, \dots, m$  in

Taylor series about  $x_n$  and collecting like terms in  $h$  and  $y$  gives;

$$L[y(x); h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y^{(2)}(x) + \dots + c_p h^p y^{(p)}(x) \quad (24)$$

**Definition:** The difference operator  $L$  associated with the discrete implicit one step method (19) are said to be of order  $p$  if in (24)

$$c_0 = c_1 = c_2 = \dots = c_{p+1} = 0, c_{p+2} \neq 0$$

**Definition :** The term  $c_{p+2}$  is called the error constant and it implies that the local truncation error is given as

$$t_{n+k} = c_{p+2} h^{(p+2)}(x_n) = O(h^{p+3})$$

**Order and Error constant of the Methods are as follows**

Applying the linear operator  $L$  (23) to determine the order and the error constants of the derived new

method gives

Expanding (19) and (20) by Taylor's series gives:  $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{256} = 0.00391$   
While the order and error constant derivative (19) are  
 $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{256} = 0.00391$   
The order and error constant of the main Predictor is

$$y_{n+3} = \frac{1}{24} \left( -7f_n + 11f_{n+1} + 34f_{n+2} + 22f_{n+\frac{1}{2}} \right) h^2 + 5y_{n+1} - 4y_{n+\frac{1}{2}}$$

with the first derivative as

$$y'_{n+3} = \frac{1}{720} (-1209f_n - 3243f_{n+1} + 2158f_{n+2} + 3914f_{n+\frac{1}{2}}) h^2 + 1440y_{n+1} - 1440y_{n+\frac{1}{2}}$$

Expanding (20) by Taylor's series gives:  $c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{256} = 0.00391$   
Hence the method is of order 4 and the Error constant = 0.00391

### Consistency

**Definition:** Given a discrete hybrid method (23), the first and second characteristics polynomials are respectively defined as

$$\begin{aligned} \rho(z) &= \sum_{j=0}^k \alpha_j z^j \\ \sigma(z) &= \sum_{j=0}^k \beta_j z^j \end{aligned} \quad (25)$$

where  $z$  is the principle root,  $\alpha \neq 0$  and  $\alpha_0^2 + \beta_0^2 \neq 0$

**Definition:** The discrete implicit hybrid method (23) is said to be consistent if it satisfies the following conditions:

- (i) the order  $p \geq 1$
- (ii)  $\sum_{j=0}^k \alpha_j = 0$
- (iii)  $\rho(1) = \rho'(1) = 0$
- (iv)  $\rho''(1) = 2!\sigma(1)$

### Consistency Of The Method

$$y_{n+3} = \frac{1}{24} \left( -7f_n + 11f_{n+1} + 34f_{n+2} + 22f_{n+\frac{1}{2}} \right) h^2 + 5y_{n+1} - 4y_{n+\frac{1}{2}} \quad (26)$$

Condition (i) is satisfied since the scheme is of order 4

Condition (ii) is satisfied since the scheme

$$\alpha_0 = 1, \alpha_1 = -5, \alpha_2 = 4; 1 - 5 + 4 = 0$$

Condition (iii) is satisfied since  $\rho(r) = r^3 - 5r + 4r^{\frac{3}{2}}$  and

$$\rho'(r) = 3r^2 - 5 + 2r^{-\frac{1}{2}}$$

when  $r = 1; \rho(1) = \rho'(1) = 0$

Condition (iv) is satisfied since  $\rho''(r) = 6r - r^{-\frac{3}{2}}$  and  $\sigma(r) = \frac{2}{192}(11r^3 + 217r^2 - 253r - 1)$

when  $r = 1, \sigma(1) = 2! * (\frac{11}{192} + \frac{217}{192} + \frac{253}{192}) = \frac{481}{192} = \frac{481}{96}$  Therefore  $\rho''(r) = 2!\sigma(r) = \frac{481}{96}$

Hence the four conditions are satisfied, the method is consistent

### Zero Stability Of The Method

$$y_{n+3} = \frac{1}{192} (-f_n + 253f_{n+1} + 217f_{n+2} + 11f_{n+3}) h^2 + 5y_{n+1} - 4y_{n+\frac{1}{2}}$$

The first characteristics polynomial is

$$\rho(r) = r^3 + 5r - 4\sqrt{r}$$

Equating to zero and solving for  $r$  gives

$r = 0, 1$ . The root  $r$  of CCSMSS method for which  $|r| = 0, 1$  is simple, hence the method is zero stable

## 4 RESULT AND DISCUSSION

This section of this research discusses how the theory is put into practice by addressing the Initial Value Problem (IVP) of a second order differential equation (ODE).

### Test Problems

#### Problem 1

Consider a slightly stiff second order problem

$$y'' = y': y(0); y'(0) = -1; h = 0.1$$

Exact Solution:  $y(x) = 1 - \exp(x)$

This problem is prevalent in engineering applications, such as vehicle dynamics, electric circuits, and fluid dynamics [Gholamtabar Lambert (1973) and Kayode and Adeyeye (2013)]. The Continuous 3-Step Method (CCSMSS) is employed to address this problem by using power series as the basis function for interpolation at grid and off-grid points.

#### Problem 2

$$y'' = 100y': y(0) = 1, y'(0) = -10; h = 0.01$$

Exact Solution:  $y(x) = \exp(-10x)$

This problem provides insights into the performance of CCSMSS for linear systems and is compared with other numerical approaches [Awari (2013)].

#### Problem 3

##### Resonance Vibration of a Machine

The resonance vibration of a machine, modeling the behavior of a stamping machine, was investigated. The differential equation governing the instantaneous position of the base is derived and solved using CCSMSS.  $My'' + ky = F(t); y(t_0) = y_0, y'(t_0) = y'_0$

**Theoretical solution:**  $y(t) = \frac{1}{10}\cos 10t + \frac{1}{200}\sin 10t - \frac{t}{20}\cos 10t$

**Source:** Kayode and Adegboro (2018)

### 4.1 Numerical Results and Error Analysis

Numerical results obtained from applying CCSMSS to the test problems are presented, and Tables 1-3 provide a detailed comparison of errors between CCSMSS and other existing methods for each problem

**Problem 1:** Table 1 provides a comprehensive analysis of errors, drawing a meticulous comparison between CCSMSS and the methodology developed by Kayode and Adeyeye in 2013. This focused comparison is centered around addressing Problem 1, with a step size of 0.1. The results unmistakably underscore the enduring superiority of CCSMSS in terms of error, demonstrating consistent

performance advantages over the established method at various data points.

Table 1: Comparison of error for new method with Kayode and Adeyeye (2013) for problem 1 with  $h = 0.1$

X	y-computed	y-exact	Error in Kayode and Adeyeye (2013)	Error in CCSMSS
0.2	-0.221402800700000	-0.22140275816017	$8.17e - 07$	$7.15e - 08$
0.3	-0.349858882300000	-0.34985880757600	$3.10e - 06$	$4.25e - 08$
0.4	-0.491824849807609	-0.49182469764127	$6.57e - 06$	$7.15e - 08$
0.5	-0.648721516077595	-0.64872127070013	$1.14e - 05$	$1.52e - 07$
0.6	-0.822119154907579	-0.82211880039051	$1.79e - 05$	$2.45e - 07$
0.7	-1.013753239128580	-1.01375270747048	$2.64e - 05$	$3.55e - 07$
0.8	-1.225541666213150	-1.22554092849247	$3.72e - 05$	$5.32e - 07$
0.9	-1.459604084880400	-1.45960311115695	$5.06e - 05$	$7.37e - 07$
1.0	-1.718283139304140	-1.71828182845905	$6.72e - 05$	$9.74e - 07$

Table 1 provides an in-depth comparison of computed errors for CCSMSS, shedding light on the precision of this approach when applied to address Problem 1. Furthermore, Figure 1 visually represents the efficiency curve, offering a graphical depiction of how effectively CCSMSS performs in relation to the problem being considered.

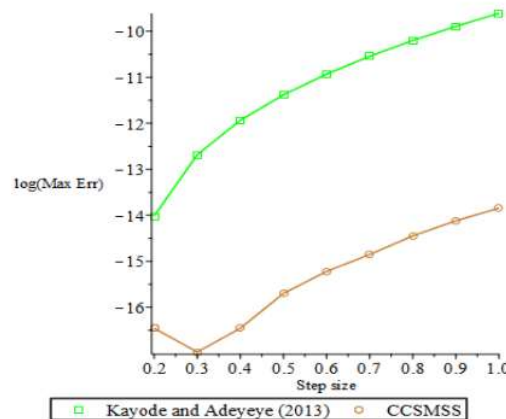


Figure 1: Efficiency curve for CCSMSS

Figure 1: Shows the efficiency curve for Kayode and Adeyeye (2013) and CCSMSS, which means that the absolute error for CCSMSS decreases more rapidly and approaching the exact solution more quickly than Kayode and Adeyeye (2013)

**Problem 2:** Comparison with Awari (2013) in this section, we delve into the comparative analysis between CCSMSS and Awari's method for addressing Problem 3, considering a step size of 0.01. Table 2 meticulously outlines the error comparison, showcasing CCSMSS's consistent demonstration of lower errors across various data points.

Table 2 serves as a comprehensive reference for understanding the error dynamics between CCSMSS and Awari's method, portraying CCSMSS's propensity for consistently lower errors. Additionally, Fig-



Table 2: Comparison of error for new method with Awari (2013) for Problem 2,  $h = 0.01$

X	y-Exact	y-Computed	Error in Awari (2013)	Error in CCSMSS
0.01	0.904837418035959573	0.904837431100000000	$1.15e - 07$	$1.30640404270e - 08$
0.02	0.818730753077981859	0.818730782300000000	$3.65e - 07$	$1.30640404270e - 08$
0.03	0.740818220681717866	0.740818270200000000	$6.05e - 07$	$4.95182821340e - 08$
0.04	0.670320046035639301	0.670320138482912654	$8.50e - 07$	$9.24472733530e - 08$
0.05	0.606530659712633424	0.670320138482912654	$1.10e - 06$	$1.38496691893e - 07$
0.06	0.548811636094026433	0.548811824875314211	$1.36e - 06$	$1.88781287778e - 07$
0.07	0.496585303791409515	0.496585561146274750	$1.45e - 06$	$2.57354865235e - 07$
0.08	0.449328964117221591	0.449329294247591180	$1.59e - 06$	$3.30130369589e - 07$
0.09	0.406569659740599112	0.406570068061162308	$1.72e - 06$	$4.08320563196e - 07$
1.00	0.367879441171442322	0.367879943917935358	$1.90e - 06$	$5.02746493036e - 07$

Figure 2 visually represents the efficiency curve, offering a graphical depiction of CCSMSS's performance in addressing Problem 2.

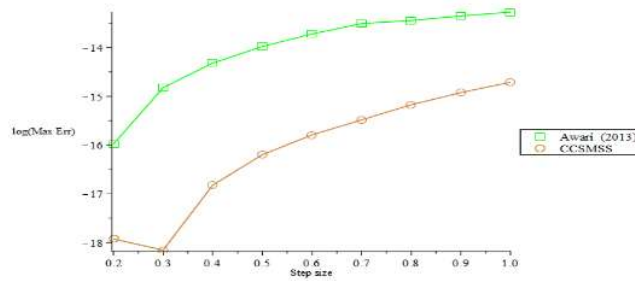


Figure 2: Efficiency curve for CCSMSS

Figure 2 shows the efficiency curve for Awari (2013) and CCSMSS, CCSMSS converges faster than Awari (2013), the solution for CCSMSS decreases exponentially with each iteration

**Problem 3:** Comparison with Kayode and Adegboro (2018) In this segment, we present a detailed examination of the error comparison between CCSMSS and Kayode and Adegboro (2018) method for addressing Problem 3, employing a step size of 0.01. Table 3 provides a comprehensive view of the error dynamics, clearly illustrating the enhanced accuracy achieved by CCSMSS.

Table 3: Comparison of error for new method with Kayode and Adegboro (2018) for Problem 3,  $h = 0.01$

X Method	y-computed	y-exact	Error in Kayode and Adegboro (2018)	Error in CCSMSS
0.01	0.99999990300017	0.99999995000015	$1.541367e - 09$	$4.6999980e - 10$
0.02	0.99999961300133	0.99999980000130	$2.976799e - 09$	$1.8699997e - 09$
0.03	0.99999912000450	0.99999954999780	$1.032860e - 09$	$1.8699997e - 09$
0.04	0.99999842800986	0.99999920001070	$2.447733e - 09$	$7.7200084e - 09$
0.05	0.99999754301899	0.99999875002085	$4.069112e - 09$	$1.20700186e - 08$
0.06	0.99999645503315	0.99999819998200	$1.337372e - 08$	$1.74494885e - 08$
0.07	0.99999516805170	0.99999755005725	$1.711910e - 08$	$2.38200555e - 08$
0.08	0.99999368807686	0.99999680008550	$1.388529e - 08$	$3.11200864e - 08$
0.09	0.99999200511012	0.99999594993950	$1.833084e - 08$	$3.94482938e - 08$
1.00	0.99999012315003	0.99999500016710	$4.286197e - 08$	$4.87701707e - 08$

Table 3 elucidates the substantial accuracy enhancement achieved by CCSMSS over Kayode and Adegboro's method for Problem 3. The associated Figure 3 further provides an insightful efficiency curve, visually capturing the performance dynamics of CCSMSS in comparison.

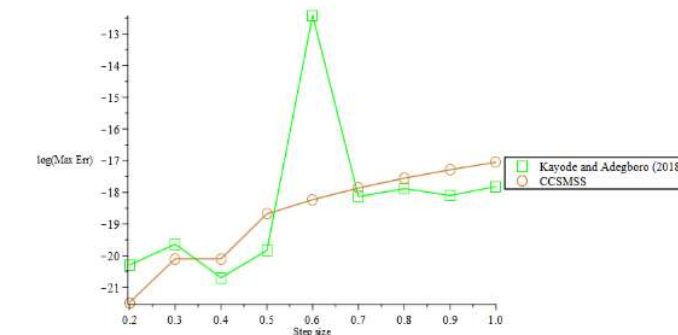


Figure 3: Efficiency curve for CCSMSS

Figure 3 shows the efficiency of the new method CCSMSS over the existing method Kayode and Adegboro(2018), as the step size decreases the more the new method approaches convergence for the types of problem

## 5 Discussion and Practical Implications

The comprehensive error analysis conducted underscores the consistent superiority of CCSMSS in achieving heightened accuracy across a diverse range of test problems when compared to existing



methods. The stability and efficiency of CCSMSS are not only supported by numerical results but are also vividly depicted through efficiency curves presented in Figures 1-3.

### **Significance in Practical Applications**

The elevated accuracy demonstrated by CCSMSS holds significant implications, particularly in the practical application of solving second-order ordinary differential equations within various engineering contexts. The method's stability and precision position it as a valuable and reliable tool for accurately simulating real-world scenarios. Researchers and practitioners can confidently utilize CCSMSS, relying on its capability to provide robust and accurate solutions in complex engineering scenarios.

### **Methodological Considerations and Limitations**

Despite the remarkable advancements demonstrated by CCSMSS, it is imperative to thoroughly acknowledge and scrutinize any associated limitations or constraints. This transparent acknowledgment offers users and researchers a nuanced perspective. Identifying potential areas for refinement or extensions is crucial for guiding future research endeavors, ensuring the continued evolution and applicability of CCSMSS. This understanding of limitations serves as a catalyst for targeted improvements, enabling the method to effectively address a broader spectrum of scenarios. This discussion not only affirms the effectiveness of CCSMSS but also emphasizes the need for continuous refinement in the pursuit of advancing numerical methodologies. The juxtaposition of achievements and limitations creates a robust foundation for further exploration, development, and application of CCSMSS in practical engineering problems.

## **6 Conclusions and Recommendations**

### **Conclusion**

This study introduces the Class of Continuous 3-Step Method (CCSMSS) as an innovative solution for addressing second-order ordinary differential equations. By employing power series as the foundational basis function and incorporating collocation and interpolation techniques, CCSMSS demonstrates notable improvements in computational efficiency. The utilization of predictor-corrector mode further refines computational processes. The implementation of CCSMSS, executed through computer programs in MAPLE software, ensures precise and reliable calculations. Thorough testing against existing methods verifies the method's superior stability and accuracy across a diverse range of problem scenarios, reinforcing its efficacy compared to prior research endeavors. The success of CCSMSS in surpassing established methods emphasizes its reliability in solving second-order ordinary differential equations. The method's distinctive ability to strike a balance between stability and accuracy positions CCSMSS as a valuable addition to the repertoire of numerical techniques for dynamic systems. This research significantly advances computational methods in the field, laying the groundwork for

future investigations and applications.

## 6.1 Recommendation

Based on the observed efficacy of the 3-Step Method, this study strongly recommends its integration for solving initial value problems in second-order ordinary differential equations. The method's effectiveness, precision, stability, and straightforward implementation make it a pragmatic choice for researchers and practitioners across diverse fields. Furthermore, the potential impact of CCSMSS extends beyond its application in engineering, suggesting promising roles in various scientific disciplines. These may encompass, but are not limited to, artificial intelligence, robotics, operational management, finance, astrophysics, computational physics, and chemistry. The incorporation of CCSMSS in these domains holds promise for transformative advancements, encouraging further interdisciplinary exploration and collaboration

## Competing Interests

The author declare that they have no competing interests.

Adegboro develop the methodology and the analysis, Ukwu arrange the write up, interpret the data and the coding. All authors have read and agreed to the published version of the manuscript

The authors thank the anonymous referees whose suggestions helped improve this manuscript.

The authors declare no conflicts of interest.

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## A

### A.1

$$a_0 = y_n \quad (27)$$

$$a_1 = -\frac{1057 h^2 f_n + 579 h^2 f_{n+1} - 249 h^2 f_{n+2} + 53 h^2 f_{n+3} + 11520 y_n - 11520 y_{n+1/2}}{5760 h} \quad (28)$$

$$a[2] = (1/2) * f[n] \quad (29)$$

$$a_3 = -1/36 \frac{11 f_n - 18 f_{n+1} + 9 f_{n+2} - 2 f_{n+3}}{h} \quad (30)$$

$$a_4 = 1/24 \frac{2 f_n - 5 f_{n+1} + 4 f_{n+2} - f_{n+3}}{h^2} \quad (31)$$

$$a_5 = -\frac{f_n - 3 f_{n+1} + 3 f_{n+2} - f_{n+3}}{120 h^3} \quad (32)$$

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