

On Solution of Scalar First Order Stochastic Differential Equations Via Semi-Implicit Milstein Method

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ABSTRACT Scalar first order stochastic differential equation (SDEs) is a mathematical equation that describes the evolution of a single random variable over time, subject to random fluctuations. It consists of deterministic part and stochastic or diffusion part. The deterministic part contains the drift function while the diffusion part contains the diffusion function. This paper examined the solution of scalar first order SDEs with regard to Balck-Scholes option pricing model (BSOPM) commonly used in financial setting. Two forms of this model have been considered, they include the form where the drift function is greater than diffusion function and the form where the diffusion function is greater than drift function. The solutions of the SDEs were calculated using semi-implicit Milstein method. Having calculated the exact and numerical solutions of the models, the absolute errors were determined. The performance of the method was compared using mean absolute error (MAE) criteria. The strong order of convergence (SOC) for the method was also calculated to determine its accuracy. To know how best the method approximates the SDEs, the SOC obtained was compared with that of two existing methods such as explicit Euler-Maruyama and Milstein methods. Graphical solutions were obtained for the method using two step-sizes.

KEYWORDS: Scalar First Order Stochastic Differential Equations, Black-Scholes Option pricing Model, Semi-Implicit Milstein Method, Mean Absolute Error, Strong Order of Convergence.

1. INTRODUCTION

In this paper, solution of scalar first order stochastic differential equation (SDE) of the form

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad X(0) = X_0, 0 \leq t \leq T \quad (1)$$

has been considered, where:

$X(t)$, $0 \leq t \leq T$ is a stochastic process, $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the drift function or the drift coefficient representing the deterministic component of the process, $g: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ is the diffusion function or diffusion coefficient representing the random component of the process and $W(t)$ is the standard Wiener process (also called Brownian motion) representing the random fluctuations. For the properties of Wiener process, see [5] and [2].

The integral form of SDE in (1) is

$$X(t) = X_0 + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s), \quad X(0) = X_0, 0 \leq t \leq T \quad (2)$$

In equation (2) the first integral is called Riemann Integral while the second integral is called Itô integral. For more information about the numerical solution of SDE in (1) see [9], [5], [1], [6], [7], [3], [4] etc.

The numerical solution of SDE in (1) was obtained using explicit Milstein method by [6] by considering varying step sizes. The aim of this paper is to determine the approximate solution of scalar first order stochastic differential equations of the form (1) via semi-implicit Milstein Method (SIMLSTM). The objectives are to find the numerical solution and the exact solution of Black-Scholes Option pricing Model (BSOPM) of two forms, first the one in which the drift function, $f(t, X(t))$ is greater than the diffusion function, $g(t, X(t))$ and second the one in which the diffusion function is greater than the drift function. Also, the approximate solution of the two aforementioned problems will be found using Mean Absolute errors (MAEs) criteria for step sizes $2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$ and 2^{-9} . The performance of the method will be obtained and compared with that of the existing methods using the criteria. Moreover, to determine the accuracy of the method being used strong order of convergence (SOC) criteria will be used. The SOC will also be compared with that of two existing methods. Using the method, graphical solution of the two problems will be obtained for step sizes 2^{-4} and 2^{-8} as considered by [6]. SIMLSTM will be compared with the explicit Euler Maruyama method by [7] and Explicit Milstein method by [6]. Comparative discussion of strong order of convergence shall be made to highlight the implications for practical applications. In our concluding remarks, we shall consider the practical implications of the research in financial modeling and real-world applications. Our applications and implications shall be restricted to financial modeling applicable in cryptocurrency operations and marketing.

In this work, we are interested in finding the approximate solution of Black Scholes Option pricing Model (BSOPM), a problem used in financial setting. The problem is of the form

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), X(t_0) = t_0 \quad (3)$$

The BSOPM in Equation (3) is a stochastic model in the form of equation (1). Two problems in the form of this model shall be considered. They are as stated below.

Problem 1

$$dX(t) = 2\lambda X(t)dt + \lambda X(t)dW(t), X(t_0) = X_0 \quad (4)$$

where $\lambda = 0.0001$ is an arbitrary value. The exact solution of the SDE in equation (4) is

$$X(t) = X_0 \exp\left((2\lambda - 0.5\lambda^2)t + \lambda W(t)\right) \quad (5)$$

Problem 1 is the Black-Scholes option pricing model with a drift function $f(t, X(t)) = 2\lambda X(t)$ and diffusion function $g(t, X(t)) = \lambda X(t)$. Here, the drift function is greater than the diffusion function.

Problem 2

$$dX(t) = \lambda X(t)dt + 2\lambda X(t)dW(t), X(t_0) = X_0 \quad (6)$$

where $\lambda = 0.0001$ is an arbitrary value.

The exact solution of the SDE in equation (6) is

$$X(t) = X_0 \exp\left((\lambda - 2\lambda^2)t + 2\lambda W(t)\right) \quad (7)$$

Problem 2 is the Black-Scholes option pricing model with a drift function $f(t, X(t)) = \lambda X(t)$ and diffusion function $g(t, X(t)) = 2\lambda X(t)$. Here, the diffusion function is greater than the drift function.

Remark 1.0

Similar problems were also used by [5], [8], [10], [6], [7] and [3]. In carrying out our numerical experiment, we shall consider the step sizes $2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$ and assume that $X_0 = 1$ for each of the problem.

II. MATERIALS AND METHODS

[6] derived one-step Milstein method and used it in determining the effect of varying stepsizes in numerical approximation of stochastic differential equations (SDEs). The Milstein method (MLSTM) is of the form

$$X_{j+1} = X_j + f(t_j, X_j) \delta t + g(t_j, X_j) \delta W_j + \frac{1}{2} g(t_j, X_j) g'(t_j, X_j) ((\delta W_j)^2 - \delta t), \quad j = 0, 1, 2, \dots, L \quad (8)$$

As stated by [11], introducing implicitness in $f(t_j, X_j) \delta t$ gives rise to drift implicit Milstein Method (DIMLTSM)

$$X_{j+1} = X_j + f(t_{j+1}, X_{j+1}) \delta t + g(t_j, X_j) \delta W_j + \frac{1}{2} g(t_j, X_j) g'(t_j, X_j) ((\delta W_j)^2 - \delta t), \quad j = 0, 1, 2, \dots, L \quad (9)$$

By expanding the third term in (9) and introducing partial implicitness gives semi-implicit Milstein method (SIMLSTM)

$$X_{j+1} = X_j + f(t_{j+1}, X_{j+1}) \delta t - \frac{1}{2} g(t_{j+1}, X_{j+1}) g'(t_{j+1}, X_{j+1}) \cdot \delta t + g(t_j, X_j) \delta W_j + \frac{1}{2} g(t_j, X_j) g'(t_j, X_j) \delta W_j^2, \quad j = 0, 1, 2, \dots, L \quad (10)$$

Where $X(t_{j+r}) := X_{j+r}$, $r = 0, 1$, $\delta W_j := (W(t_{j+r}) - W(t_{j+r-1}))$, $r = 1$, and $\delta t := (t_{j+r} - t_{j+r-1})$, $r = 1$ in equations (8) through (10).

As proposed by [8], for an autonomous SDE of the form

$$dX = f(X(t)) dt + g(X(t)) \cdot dW(t), \quad X(t_0) = X_0 \quad (11)$$

The MILSTM is of the form

$$X_{j+1} = X_j + f(X_j) \delta t + g(X_j) \delta W + \frac{1}{2} g(X_j) g'(X_j) ((\delta W_j)^2 - \delta t), \quad j = 0, 1, 2, \dots, L \quad (12)$$

Hence, the SIMLSTM for autonomous SDE becomes

$$X_{j+1} = X_j + f(X_{j+1}) \delta t - \frac{1}{2} g(X_{j+1}) g'(X_{j+1}) \cdot \delta t + g(X_j) \delta W_j + \frac{1}{2} g(X_j) g'(X_j) \delta W_j^2, \quad j = 0, 1, 2, \dots, L \quad (13)$$

The Method Generated from the Application of SIMLSTM to Problem 1 in the Form of Scalar Stochastic Differential Equation.

Application of SIMLSTM of equation (10) to problem 1 gives:

$$X_{j+1} = \frac{X_j + \lambda X_j \delta W_j + 0.5 \lambda^2 X_j \delta W_j^2}{1 - (2\lambda - 0.5 \lambda^2) \delta t}, \quad j = 0, 1, 2, \dots, L$$

Or

$$X_{j+1} = \frac{X_j (1 + \lambda(\delta W_j + 0.5\lambda\delta W_j^2))}{1 - \lambda(2 - 0.5\lambda)\delta t}, \quad j = 0, 1, 2, \dots, L \quad (14)$$

The Method Generated from the Application of SIMLSTM to problem 2 in the Form of Scalar Stochastic Differential Equation.

Application of SIMLSTM of equation (10) to problem 2 gives

$$X_{j+1} = \frac{X_j + 2\lambda X_j \delta W_j + 2\lambda^2 X_j \delta W_j^2}{1 - \lambda\delta t + 2\lambda^2\delta t}, \quad j = 0, 1, 2, \dots, L$$

Or

$$X_{j+1} = \frac{X_j(1 + 2\lambda\delta W_j + 2\lambda^2\delta W_j^2)}{1 - \lambda\delta t(1 - 2\lambda)}, \quad j = 0, 1, 2, \dots, L \quad (15)$$

Our interest is to consider SIMLSTM in (10) to find solution of scalar first order non-autonomous SDE of the form (1).

Remark 2.0

The SDE in equation (1) will be solved by applying equation (10) using discretized interval $[0, T]$ as $0 < t_0 < t_1 < \dots < t_j < t_{j+1} = T$. Let $\delta t = \frac{T}{L}$ be the stepsize defined as $\delta t := t_{j+1} - t_j$, where N are some integer and $t_j = j\delta t$. The δt -space path increment $\delta W_j := W_j - W_{j-1}$ will be approximated by summing the underlying dt -space increments as established by [5] using

$$Winc = \text{sum}(dW(R*(j-1)+1 : R*j)) \quad (16)$$

Wiener increment dW will be generated in MATLAB over the space intervals by using $dW := \text{sqrt}(dt)*\text{rand}(1, N)$. For computational purpose, we shall assume that $R = 1$, $Dt = R*dt$ and $L = N/R$. Exact and numerical solution will be obtained using MATLAB software program.

In this paper, the exact and numerical solution of the given SDE will be represented by $X(t)$ and $X^h(t)$ respectively, where $h = \delta t$ is the stepsize. As in [3], absolute error will be defined as

$$ABS_{error} = |X(t) - X^h(t)| \quad (17)$$

The mean absolute error (MAE) will be defined similarly by

$$E(ABS_{error}) = E(|X(t) - X^h(t)|) \quad (18)$$

As defined by [3], a method is said to have strong order of convergence (SOC) λ if there exist a constant C such that

$$E(|X(t) - X^h(t)|) \leq Ch^\lambda \quad (19)$$

The mean absolute error in equation (18) will be useful in determining the SOC λ . The constant C in equation (19) is referred to as residual. The strong order of convergence λ and the residual C will be obtained by using least squares fit established by [5] as stated by [3]. The accuracy of the methods will be determined by strong order of convergence criteria. The effect of varying time-increment δt will also be assessed by using $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$. MATLAB software shall be used to code the program for the implementation.

We can now determine the numerical solution of the given problems in (4) and (6) using semi-implicit Milstein method (SIMLTM).

III. RESULTS AND DISCUSSION

Approximate Solution of Scalar Stochastic Differential Equations Using Semi-Implicit Milstein Methods (SIMLSTM) for Problem 1.

Table 1 shows the results of application of SIMLSTM of equation (14) to Problem 1 in equation (4). It shows the exact solution and numerical solution of problem 1 using the method with step size 2^{-4} . It can be observed that the mean absolute error (MAE) is $4.297065259173394e-010$.

Table 1: Result of Using SIMLSTM (14) for Problem 1 with Step size $h=2^{-4}$

t-value	Exact Solution	Numerical Solution	Absolute Error
0.062500	1.000035213412162	1.000035213490285	7.81226195e-011
0.125000	0.999992194240297	0.999992194396566	1.56268665e-010
0.187500	0.999998717493128	0.999998717727519	2.34391395e-010
0.250000	1.000012935644641	1.000012935957158	3.12517123e-010
0.312500	0.999974930990167	0.999974931380813	3.90646515e-010
0.375000	0.999978328502518	0.999978328971286	4.68768024e-010
0.437500	0.999988796767910	0.999988797314804	5.46893753e-010
0.500000	0.999951804297058	0.999951804922069	6.25011598e-010
0.562500	0.999984007130832	0.999984007833983	7.03150982e-010
0.625000	1.000014922754158	1.000014923535452	7.81294585e-010

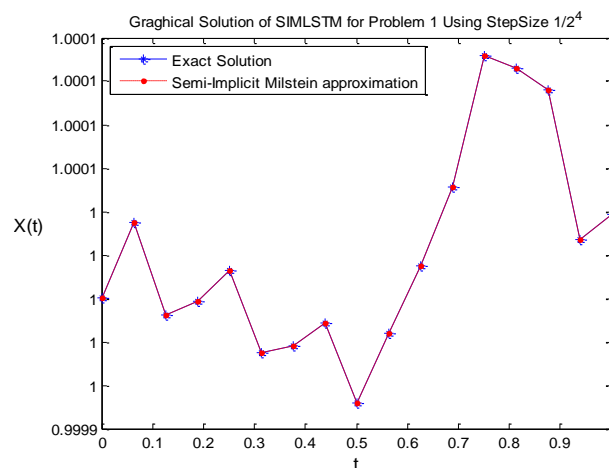


Figure 1: The sample path of the exact solution and numerical solution of problem 1 using **SIMLSTM** with stepsize 2^{-4} .

Table 2 shows the results of application of SIMLSTM of equation (14) to Problem 1 in equation (4). It shows the exact solution and numerical solution of problem 1 using the method with stepsize 2^{-8} . It can be observed that the mean absolute error (MAE) is $2.685902611432312e-011$.

Table 2: Result of using SIMLSTM (14) for solution of Problem 1 with $h=2^{-8}$

t-value	Exact Solution	Numerical Solution	Absolute Error
0.062500	0.999972159883026	0.999972159887911	4.88531438e-012
0.125000	0.999978280985938	0.999978280995705	9.76763115e-012
0.187500	1.000027372023912	1.000027372038562	1.46500589e-011
0.250000	1.000065836108020	1.000065836127553	1.95328198e-011
0.312500	1.000101764246374	1.000101764270790	2.44162468e-011
0.375000	1.000108726423491	1.000108726452790	2.92994518e-011
0.437500	1.000148957998176	1.000148958032359	3.41833228e-011
0.500000	1.000172569073211	1.000172569112279	3.90671939e-011
0.562500	1.000206392121796	1.000206392165748	4.39521752e-011
0.625000	1.000193604241676	1.000193604290512	4.88360463e-011

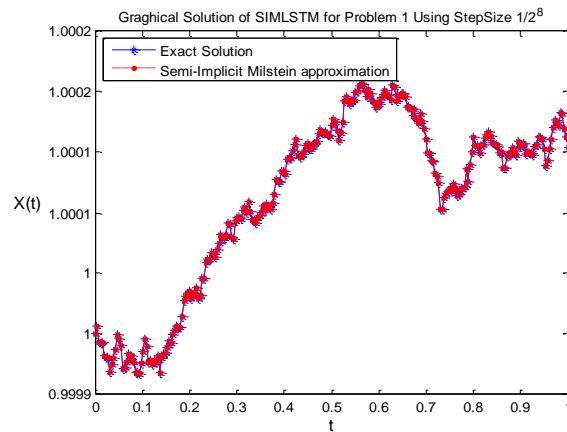


Figure 2: The sample path of the exact solution and numerical solution of problem 1 using SIMLSTM with stepsize 2^{-8} .

Remark 3.0

The exact solution and numerical solution of problem 1 using SIMLSTM with other stepsizes 2^{-5} , 2^{-6} , 2^{-7} , 2^{-9} were also obtained. The absolute errors were also calculated. Moreover, the mean absolute errors $E(ABS_{error})$ were then determined. The $E(ABS_{error})$ can be observed in Table 5 for each stepsize.

Approximate Solution of Scalar Stochastic Differential Equations Using Semi-Implicit Milstein Methods (SIMLSTM) for Problem 2.

Table 3 shows the results of application of SIMLSTM of equation (15) to Problem 2 in equation (6). It shows the exact solution and numerical solution of problem 2 using the method with stepsize 2^{-4} . It can be observed that mean absolute error (MAE) is $1.077019140538482e-010$.

Table 3: Result of using SIMLSTM (15) for solution of Problem 2 with Stepsize $h=2^{-4}$

t-value	Exact Solution	Numerical Solution	Absolute Error
0.062500	1.000051676294542	1.000051676314052	1.95090610e-011
0.125000	0.999946888580125	0.999946888619382	3.92574862e-011
0.187500	0.999941184839290	0.999941184898070	5.87798699e-011
0.250000	0.999950869828879	0.999950869907182	7.83031417e-011
0.312500	0.999856118578793	0.999856118676781	9.79882842e-011
0.375000	0.999844164928923	0.999844165046431	1.17508558e-010
0.437500	0.999846350840548	0.999846350977577	1.37029388e-010
0.500000	0.999753631624889	0.999753631781586	1.56696989e-010
0.562500	0.999799278528140	0.999799278704353	1.76213377e-010
0.625000	0.999842351463466	0.999842351659199	1.95732985e-010

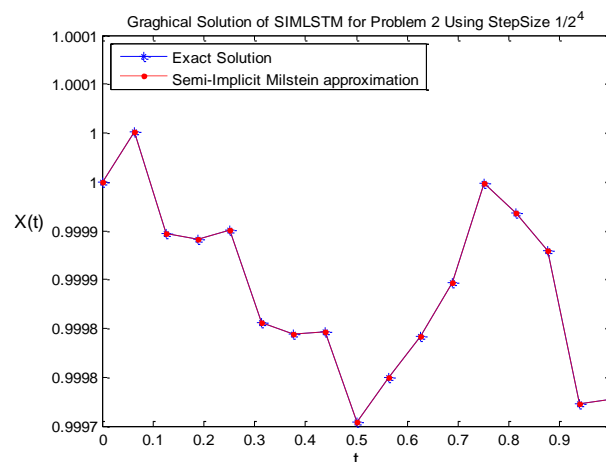


Figure 3: The Sample Path of the Exact Solution and Numerical Solution of Problem 2 using SIMLSTM with Step Size 2^{-4} .

Table 4 shows the results of application of SIMLSTM of equation (15) to Problem 2 in equation (6). It shows the exact solution and numerical solution of problem 2 using the method with stepsize 2^{-8} . It can be observed that mean absolute error (MAE) is $6.716482925384070e-012$.

Table 4: Result of using SIMLSTM (15) for Solution of Problem 2 with Stepsize $h=2^{-8}$

t-value	Exact Solution	Numerical Solution	Absolute Error
0.062500	0.999925571135931	0.999925571137167	1.23590027e-012
0.125000	0.999919063525686	0.999919063528143	2.45736764e-012
0.187500	0.999998491424748	0.999998491428423	3.67517128e-012
0.250000	1.000056666987349	1.000056666992239	4.88964425e-012
0.312500	1.000109771036888	1.000109771042994	6.10533846e-012
0.375000	1.000104942782512	1.000104942789838	7.32569561e-012
0.437500	1.000166653320192	1.000166653328735	8.54383231e-012
0.500000	1.000195122403663	1.000195122413425	9.76174697e-012

0.562500	1.000244015793755	1.000244015804731	1.09752207e-011
0.625000	1.000199684689921	1.000199684702116	1.21949117e-011

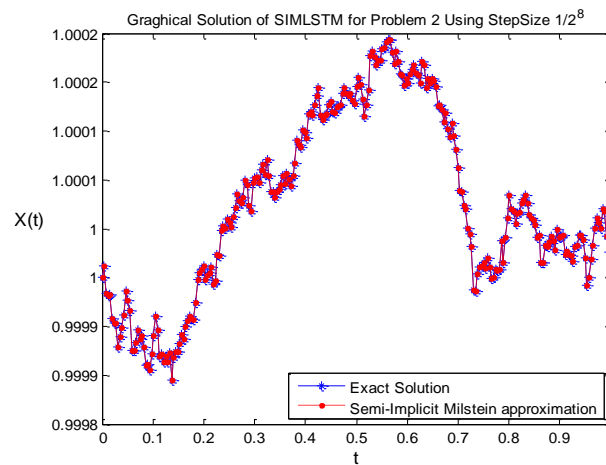


Figure 4: The Sample Path of the Exact Solution and Numerical Solution of Problem 2 using SIMLSTM with Step Size 2^{-8} .

Remark 3.1

The exact solution and numerical solution of problem 2 using SIMLSTM with other stepsizes 2^{-5} , 2^{-6} , 2^{-7} , 2^{-9} were also obtained. The absolute errors were also calculated. Moreover, the mean absolute errors $E(ABS_{error})$ were then determined. The $E(ABS_{error})$ can be observed in Table 6 for each stepsize.

Comparison in Performance of Semi-Implicit Milstein Method (SIMLSTM) Applied to Problem 1 with EEMM (an Existing Method) Using Mean Absolute Error (MAE) Criteria.

Table 5 below shows the comparison in performance of SIMLSTM with EEMM Method using Problem 1. It can be observed from table 5 below that the MAE obtained by applying SIMLSTM to problem 1 for each stepsize is less than the corresponding stepsize obtained using EEMM for problem 1 in [7] with $\lambda = 0.0001$. This shows that SIMLSTM performs better than EEMM.

Table 5. Comparison in Performance of SIMLSTM with Explicit Euler Maruyama Method (EEMM) by [7] Using Problem 1.

STEP-SIZE	SIMLSTM for Problem 1 MAE ($\lambda = 0.0001$)	EEMM for Problem 1 [7], MAE ($\lambda = 0.0001$)
2^{-4}	4.297065259173394e-010	6.366694393911132e-010
2^{-5}	2.148616173869300e-010	1.049213882442501e-009
2^{-6}	1.074331290595865e-010	9.695811487020478e-010
2^{-7}	5.371942890519676e-011	6.307070421485150e-010
2^{-8}	2.685902611432312e-011	2.521636677244032e-010
2^{-9}	1.343090083594234e-011	1.147927086719847e-010

Comparison in Performance of Semi-Implicit Milstein Method (SIMLSTM) Applied to Problem 1 with EMLSTM (an Existing Method) Using Mean Absolute Error (MAE) Criteria.

Table 6 shows the comparison in performance of SIMLSTM with EMLSTM Method using Problem 1. It can be observed from table 6 below that the MAE obtained by applying SIMLSTM to problem 1 for each stepsize is less

than the corresponding stepsize obtained using EMLSTM for problem 1 in [6] with $\lambda = 0.0001$. This shows that SIMLSTM performs better than EMLTSM.

Table 6. Comparison in Performance of SIMLSTM with Explicit Milstein Method (EMLSTM) by [6] Using Problem 1

STEP-SIZE	SIMLSTM for Problem 1 MAE ($\lambda = 0.0001$)	EMLSTM for Problem 1 [6], MAE ($\lambda = 0.0001$)
2^{-4}	4.297065259173394e-010	5.873183162030670e-010
2^{-5}	2.148616173869300e-010	2.219070815989710e-010
2^{-6}	1.074331290595865e-010	1.084975331799853e-010
2^{-7}	5.371942890519676e-011	8.610868684755246e-011
2^{-8}	2.685902611432312e-011	6.004124974978709e-011
2^{-9}	1.343090083594234e-011	1.823129425204684e-011

Comparison in Performance of Semi-Implicit Milstein Method (SIMLTM) Applied to Problem 2 with EEMM (an Existing Method) Using Mean Absolute Error (MAE) Criteria.

Table 7 shows the comparison in performance of SIMLSTM with EEMM Method using Problem 2. It can be observed from table 7 below that the MAE obtained by applying EEMM to problem 2 for each stepsize is less than the corresponding stepsize obtained using EEMM for problem 2 in [7] with $\lambda = 0.0001$. This shows that SIMLSTM outperform EEMM.

Table 7. Comparison in Performance of SIMSTM with Explicit Euler Maruyama Method (EEMM) by [7] Using Problem 2.

Stepsize	SIMLSTM $\lambda = 0.0001$ for Problem 2	EEMM $\lambda = 0.0001$ for Problem 2 [7]
2^{-4}	1.077019140538482e-010	3.768894007993850e-009
2^{-5}	5.389303447955740e-011	4.661047592335876e-009
2^{-6}	2.696188827755464e-011	4.122836083642057e-009
2^{-7}	1.346840416971418e-011	2.418578415319672e-009
2^{-8}	6.716482925384070e-012	9.091794761495464e-010
2^{-9}	3.366373846347415e-012	4.211989312885578e-010

Comparison in Performance of Semi-Implicit Milstein Method (SIMLTM) Applied to Problem 2 with EMLSTM (an Existing Method) Using Mean Absolute Error (MAE) Criteria.

Table 8 shows the comparison in performance of SIMLSTM with EMLSTM Method using Problem 2. It can be observed from table 8 below that the MAE obtained by applying SIMLSTM to problem 2 for each stepsize is less than the corresponding stepsize obtained using EMLSTM for problem 2 in [6] with $\lambda = 0.0001$. This shows that SIMLSTM outperform EMLTSM.

Table 8. Comparison in Performance of SIMSTM with EMLSTM by [6] Using Problem 2.

Stepsize	SIMLSTM $\lambda = 0.0001$ for Problem 2	EMLSTM $\lambda = 0.0001$ for Problem 2 [6]
2^{-4}	1.077019140538482e-010	8.979161303379613e-010
2^{-5}	5.389303447955740e-011	3.698142569952268e-010
2^{-6}	2.696188827755464e-011	1.891228174066839e-010
2^{-7}	1.346840416971418e-011	7.224950637763072e-011
2^{-8}	6.716482925384070e-012	4.293621014284099e-011
2^{-9}	3.366373846347415e-012	1.164781604501286e-011

Comparison in Strong Order of Convergence (SOC) of Semi-Implicit Milstein Method with EEMM Using Problem 1.

Table 9 shows the comparison in performance of SIMLSTM with EMLSTM Method using Problem 2. It can be observed from table 9 that the SOC of SIMLSTM which is 1.0000 is greater than that of EEMM by [7] which is 0.5471 and EMLSTM by [6] which is 0.8868. This shows that SIMLSTM outperformed the two other methods in terms of accuracy.

Table 9. Comparison in Strong Order of Convergence (SOC) and Residual of SIMLSTM with EEMM by [7] and EMLSTM by [6] Using Problem 1.

Stepsize	SOC of Problem 1	Residual of Problem 1
SIMLSTM	1.0000	5.3912e-005
EEMM [7]	0.5471	1.0975
EMLSTM [6]	0.8868	5.7386e-001

Comparison in SOC of SIMLSTM with EMLSTM Using Problem 2

Table 10 shows the comparison in strong order of convergence of SIMLSTM with EEMM and EMLSTM. It can be observed from table 10 that the SOC of EMLSTM by [6] which is 1.2014 is greater than that of EEMM by [7] that is 0.6758 and SIMLSTM which is 1.0004. This is a contradiction as we claimed in sections 3.5 and 3.6 using tables 7 and 8 that SIMLSTM outperformed the two other methods. The low accuracy obtained here using SIMLSTM can be attributed to what happens in practical application. Low interest rate $r = \lambda$ and high volatility $\sigma = 2\lambda$ has greater effect on Marketers behaviour in financial modeling. The target of every Marketer is to make profit. When the Marketers are chasing low interest rate, many of them disappear from the market. This hitherto causes instability in the market.

Table10. Comparison in Strong Order of Convergence (SOC) and Residual of SIMLSTM with EEMM by [7], and EMLSTM by [6] Using Problem 2

Stepsize	SOC of Problem 2	Residual of Problem 2
SIMLSTM	1.0004	2.5254e-003
EEMM [7]	0.6758	9.409 e-001
EMLSTM [6]	1.2014	3.5921e-001

IV. CONCLUSION

In this paper, efforts have been geared towards determining the solution of scalar first order stochastic differential equations (SDEs) commonly used in financial settings. The stochastic models considered include the form in which the drift function is greater than the diffusion function and the form in which the diffusion function is greater than the drift function. The solutions were obtained using semi-implicit Milstein method (SIMLSTM). In determining the performance of the method, absolute errors were calculated for six stepsizes $2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$ by calculating the difference between the exact solution and numerical solution using ten iterative values. Mean absolute errors (MAEs) were then calculated for problems 1 and problem 2. The MAEs for problems 1 and 2 were compared with that of the existing methods such as explicit Euler-Maruyama Method (EEMM) by [7] and explicit Milstein method (EMLSTM) by [6]. Accuracy of the method was determined by calculating strong order of convergence (SOC) using MAEs for each problem. SOC obtained was compared with that of the existing methods aforementioned. Graphical solutions for the two given problems were obtained for the stepsizes $2^{-4}, 2^{-8}$.

Having examined the MAE, the result showed that in general as the stepsizes decreases iteratively the MAE decreases. In addition, by comparing the MAE, the result showed that MAE for each stepsize using SIMLSTM is

less than corresponding MAE for each stepsize using EEMM by [7] and EMLSTM by [6]. Using MAE criteria, observation showed that SIMLSTM outperformed the two existing methods.

The accuracy of the method was determined by calculating strong order of convergence (SOC) using MAE obtained for problems 1 and 2. Having compared SOC obtained for SIMLSTM with that of the two existing methods using problem 1, the results showed that SIMLSTM outperform EEMM and EMLSTM since SOC of SIMLSTM is 1.000 which is greater than that of EEMM which is 0.5471 and that of EMLSTM which is 0.8868.

Having compared SOC obtained for SIMLSTM with the two existing methods using problem 2, the results showed that EMLSTM outperform SIMLSTM and EEMM since SOC of EMLSTM is 1.2014 which is greater than that of SIMLSTM which is 1.0004 and that of EEMM which is 0.6758. The low accuracy of SIMLSTM can be attributed to what happens in real life application of Black-Scholes option pricing provided as follows. Observation showed in problem 1 that the interest rate $r = 2\lambda$ is greater than the volatility $\sigma = \lambda$. This expresses the accuracy of SIMLSTM using SOC criteria as it performed better than the other two methods. However, in problem 2 the interest rate $r = \lambda$ is less than the volatility $\sigma = 2\lambda$. This expresses the low accuracy of SIMLSTM using SOC criteria.

In Black-Scholes option pricing, allowing the interest rate r to be greater than volatility σ (i.e. $r > \sigma$) as applicable to problem 1 has several implications one of which is underpricing of options. When $r > \sigma$, option price may become underpriced, as the model would imply lower expected returns from volatility than from the risk-free interest rate. To address the above implication, practitioners may consider using alternative models such as stochastic volatility model that can better capture the complex relationships between interest rate and volatility. Moreover, allowing the volatility σ to be greater than interest r (i.e. $\sigma > r$) as applicable to problem 2 has several implications one of which is overpricing of options. When $\sigma > r$, option prices may become overpriced, as the model would imply higher expected returns from volatility than from risk-free interest rate. To address the implications, practitioners often use alternative models that allow volatility to be stochastic, rather than constant, which can better capture real world dynamics. Furthermore, in marketing, particularly in cryptocurrency operations, allowing volatility to be greater than interest rate has several positive implications, one of which is increased trading activity, as investors capitalize on price movements. It also has negative implications, one of which is increase price risk. Allowing $\sigma > r$ especially in cryptocurrency operations can lead to increase price speculation, potentially driving up prices and creating bubbles. To navigate the aforementioned implication, marketers and cryptocurrency operators should consider implementing risk management strategies such as stop-loss orders, position sizing and diversification. In addition, educational resources can be provided to help investors understand the risks and opportunities associated with high volatility. Finally, allowing interest rate to be greater than volatility has positive implications one of which is increase investment attractiveness. This can make cryptocurrency investments more attractive to investors, potentially leading to increased adoption and demand. It also has negative implications, one of which is increased price volatility which exacerbate cryptocurrency price volatility, making it more challenging for investors to predict market movements. To navigate the implication, marketers and cryptocurrency operators should carefully consider the potential effects of high interest rates on their operations and investor behaviour.

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