

Hyers-Ulam-Rassias Stability of Certain Classes of Nonlinear Functional Differential Equations of Third Order

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ABSTRACT This paper investigates the Hyers-Ulam-Rassias stability of specific classes of third-order nonlinear functional differential equations, building on the foundational concepts introduced by Ulam and Hyers. The primary objective is to derive stability conditions and Hyers-Ulam-Rassias constants for these equations. Using Bihari integral inequality and Gronwall-Bellman-Bihari type inequalities as the methodological framework, the stability results are rigorously established. The findings are illustrated through practical examples, demonstrating the applicability of the derived stability conditions to problems in fields such as hereditary dynamics, population growth models, and the spread of infectious diseases. These results contribute to both the theoretical understanding and practical application of stability analysis in nonlinear differential equations.

KEYWORDS: Hyers-Ulam-Rassias stability, nonlinear differential equation, integral inequality, Gronwall-Bellman-Bihari type inequality, stability

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I. INTRODUCTION

The study of stability problem for various functional equations originated from a famous talk given by S.M. Ulam. In 1940, Ulam[51] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function exist". This problem was also put in this sense "For what metric group G is it true that an ϵ automorphism of G is necessarily near to a strict automorphism"? The problem for approximately additive mappings, on Banach spaces, was solved by Hyers[19]. After then, many Mathematicians have extensively investigated the stability problem of functional equations see[3, 6, 7, 8, 9, 17, 20, 25, 26, 35, 38, 48, 50]. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [36, 37]. Thereafter, Hyers-Ulam stability has been proved for the first order linear differential equations by several Mathematicians such as [4, 22, 23, 24, 31, 32, 46, 49]. In addition, the following authors proved the Hyers-Ulam stability of the second, third and n th order linear ordinary differential equations[1, 2, 18, 27, 28, 29, 33, 40, 44, 47, 52]. Recently authors such as Rus[43, 42], Qarawani[39, 40], Quasausy[41], Ravi[44] and Fakunle and Arawomo[11, 12, 13, 14, 15, 16] considered the Hyers-Ulam stability of the first and second order nonlinear differential equations.

In this paper, we study the Hyers-Ulam-Rassias stability of the following classes of nonlinear third order differential equations:

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) = G(t, u(t), u'(t)) \quad (1)$$

$$u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)r(u(t)) = G(t, u(t), u'(t)) \quad (2)$$

$$u'''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) = G(t, u(t), u'(t)) \quad (3)$$

and

$$u'''(t) + np(t)Q(t, u(t))u'(t) + f(t)\rho(u(t)) = H(t, u(t), u'(t)) \quad (4)$$

with initial conditions

$$u(t_0) = u'(t_0) = u''(t_0) = 0, \quad t > 0. \quad (5)$$

where $n \in \mathbf{N}$ (the set of natural numbers), $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$, $g(t_0, 0) = 0$, $Q(t_0, 0) = 0$, $G(t_0, 0, 0) = 0$, $H(t_0, 0, 0) = 0$, $\rho, g, f \in C(\mathbf{R}_+, \mathbf{R}_+)$, $Q \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$, $G, H \in C(\mathbf{I} \times \mathbf{R}^2, \mathbf{R})$, $p(t), \alpha(t), \beta(t), r(t) \in C(\mathbf{I}, \mathbf{R}_+)$, $\mathbf{I} = (0, \infty)$, $\mathbf{R}_+ = [0, \infty)$. The case $G(t, u(t), u'(t)) = 0$ and $H(t, u(t), u'(t)) = 0$ is considered also.

II. PRELIMINARIES

The following definitions, lemmas and theorems are necessary for our results and discussions.

Definition 1. Equation (1) is Hyers-Ulam-Rassias stable if for every positive function $\varphi(t)$ defined as $\varphi(t): \mathbf{I} \rightarrow \mathbf{R}_+$, if there exists a constant $C_\varphi > 0$ with the following properties: for every $t \in \mathbf{I}$ sufficiently large and solution $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) - G(t, u(t), u'(t))| \leq \varphi(t) \quad (6)$$

then, there exists any solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ of equation (1) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in \mathbf{I},$$

where C_φ is Hyers-Ulam-Rassias constant.

Definition 2. Equation (2) is Hyers-Ulam-Rassias stable if for every positive function $\varphi(t)$ defined as $\varphi(t): \mathbf{I} \rightarrow \mathbf{R}_+$, if there exists a constant $C_\varphi > 0$ with the following properties: for every $t \in \mathbf{I}$ sufficiently large and solution $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)r(u(t)) - G(t, u(t), u'(t))| \leq \varphi(t) \quad (7)$$

then, there exists any solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ of equation (2) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in \mathbf{I},$$

where C_φ is Hyers-Ulam-Rassias constant.

Definition 3. Equation (3) is Hyers-Ulam-Rassias stable if for every positive function $\varphi(t)$ defined as $\varphi(t): \mathbf{I} \rightarrow \mathbf{R}_+$, if there exists a constant $C_\varphi > 0$ with the following properties: for every $t \in \mathbf{I}$ sufficiently large and solution $u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ satisfying

$$|u'''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) - G(t, u(t), u'(t))| \leq \varphi(t) \quad (8)$$

then, there exists any solution $u_0(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$ of equation (3) such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in \mathbf{I},$$

where C_φ is Hyers-Ulam-Rassias constant.

Definition 4. The differential equation (4) has the Hyers-Ulam-Rassias stability with initial condition (5), if there exists a positive constant C_φ with the following property: for every $u(t) \in C^3(\mathbf{R}_+)$, which satisfies

$|u'''(t) + np(t)Q(t, u(t))u'(t) + r(t)\rho(u(t)) - H(t, u(t), u'(t))| \leq \varphi(t)$, (9)
then there exists a function $u_0(t) \in C^3(\mathbf{R}_+)$ satisfies (4) with initial conditions (5) such that
 $|u(t) - u_0(t)| \leq C_\varphi \varphi(t)$,
we call such C_φ Hyers-Ulam-Rassias constant for the differential equation (4).

Theorem 2.1 [30, 34] (Generalised First Mean Valued Theorem) .

If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\int_{t_0}^t g(s)f(s)ds = g(\xi) \int_{t_0}^t f(s)ds$.

Definition 5. A function $\omega: [0, \infty) \rightarrow [0, \infty)$ is said to belong to a class Ψ if

1. $\omega(u)$ is nondecreasing and continuous for $u \geq 0$,
2. $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$,
3. there exist a function ϕ , continuous on $[0, \infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Lemma1 [5] Let $u(t)$, $f(t)$ be positive continuous functions defined on $a \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M \int_{t_0}^t f(s)\omega(u(s))ds, \quad t_0 \leq t < b. \quad (10)$$

Implies the inequality

$$u(t) \leq \Omega^{-1} \left(\Omega(k) + M \int_{t_0}^t f(s)ds \right), \quad t_0 \leq t \leq b' \leq b, \quad (11)$$

where

$$\Omega(u) = \int_{u_0}^u \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \quad (12)$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M \int_{t_0}^t f(s)ds \in \text{Dom}(\Omega^{-1}).$$

Theorem 2.2. [10] Let

1. $u(t), r(t): (0, \infty) \rightarrow (0, \infty)$ and continuous on $(0, \infty)$,
2. $\varpi \in \Psi$,
3. $n > 0$ be monotonic, nondecreasing and continuous on $(0, \infty)$,

if

$$u(t) \leq n(t) + \int_0^t f(s)\varpi(u(s))ds, \quad 0 < t < \infty, \quad (13)$$

then

$$u(t) \leq n(t)\Omega^{-1} \left(\Omega(1) + \int_0^t f(s)ds \right) \quad 0 < t \leq b, \quad (14)$$

where $(0, b) \subset (0, \infty)$, where $\Omega(u)$ is defined in (12) and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0, b)$ is so chosen that

$$\Omega(1) + \int_0^t f(s)ds \in \text{Dom}(\Omega^{-1}).$$

Corollary 1 [15, 16] Suppose $\rho(t)$ is a nonnegative, monotonic, nondecreasing continuous function on \mathbf{R}_+ . Let

$$u(t) \leq \rho(t) + T \int_{t_0}^t r(s)\beta(u(s))ds + L \int_{t_0}^t h(s)\varpi(u(s))ds, \quad (15)$$

for T and L be positive constants, then

$$u(t) \leq \rho(t)\Omega^{-1}\left(\Omega(1) + L \int_{t_0}^t h(s)\varpi\left(F^{-1}\left(F(1) + T \int_{t_0}^t r(\alpha)d\alpha\right)\right)ds\right) \\ F^{-1}\left(F(1) + T \int_{t_0}^t r(s)ds\right), \quad t \in \mathbf{I}, \quad (16)$$

where

$$F(u) = \int_{u_0}^u \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u, \quad (17)$$

and $\Omega(u)$ is defined as (12).

Theorem 2.3. [15, 16] If $u(t), r(t), h(t), \rho(t), g(t) \in \mathcal{C}(\mathbf{R}_+)$ and $\omega, f, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ be submultiplicative. If

$$u(t) \leq \rho(t) + A \int_{t_0}^t r(s)\beta(u(s))ds + B \int_{t_0}^t h(s)\varpi(u(s))ds + \\ L \int_{t_0}^t g(s)\gamma(u(s))ds \quad (18)$$

for $A, B, L > 0$, then

$$u(t) \leq \rho(t)Y^{-1} \\ \left[Y(1) + L \int_{t_0}^t g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B \int_{t_0}^s h(\alpha)\varpi(T(\alpha))d\alpha\right)T(s)\right]ds\right] \\ \Omega^{-1}\left(\Omega(1) + B \int_{t_0}^t h(s)\varpi(T(s))ds\right)T(t) \quad (19)$$

where $T(t)$ is given as

$$T(t) = F^{-1}\left(F(1) + A \int_{t_0}^t r(s)ds\right) \quad (20)$$

and

$$Y(r) = \int_{r_0}^r \frac{ds}{\gamma(s)}, \quad 0 < r_0 \leq r, \quad (21)$$

and F^{-1} , Ω^{-1} and Y^{-1} are the inverses of F , Ω , Y respectively $t \in (0, b) \subset (I)$. So that

$$Y(1) + L \int_{t_0}^t g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B \int_{t_0}^s h(\alpha)\varpi(T(\alpha))d\alpha\right)T(s)\right]ds \in \text{Dom}(Y^{-1})$$

III. RESULTS AND DISCUSSION

In this section we investigate the Hyers-Ulam-Rassias stability of differential equations (1),(2),(3),(4) and consider the case when $G(t, u(t), u'(t)) = 0$ and $H(t, u(t), u'(t)) = 0$

Theorem 3.1 .Equation (1) is stable in the sense of Hyers-Ulam-Rassias stability, if the following prescribed conditions are satisfied:

1. let $D(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds < 0$,
2. $|G(t, u(t), u'(t))| \leq \phi(t)\varpi(|u(t)|)(|u'(t)|)^n$, where n a positive integer, ϖ belongs to class Ψ , $\phi(t) \in \mathbf{R}_+$,
3. let $|u'''(s)|ds \leq L$, where L , $|u'(t)| \leq \lambda$, $\lambda > 0$, and $|u''(t)| \leq k_1$, $k_1 > 0$, $\int_{t_0}^t |u'''(s)|ds \leq \delta$, where $\delta > 0$,
4. $\lim_{t \rightarrow \infty} \int_{t_0}^t \phi(s)ds \leq k_2$, where $k_2 > 0$, $\lim_{t \rightarrow \infty} \int_{t_0}^t \rho(s)ds \leq k_3$, where $k_3 > 0$,
5. $\lim_{t \rightarrow \infty} \int_{t_0}^t \beta(s)ds \leq k_4$, where $k_4 > 0$, $\lim_{t \rightarrow \infty} \int_{t_0}^t \alpha(s)ds \leq k_5$, where $k_5 > 0$,
5. there exists $\sigma > 0$ such that $\int_{t_0}^t \varphi(s)ds \leq \sigma\varphi(t)$,

and Hyers-Ulam-Rassias constant is given as

$$C_{\varphi_1} = \frac{(1+\delta)}{\tau} Y^{-1} \left[Y(1) + k_3 \frac{\lambda^{n+1}}{\tau} \varpi \left[\Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^* \right] \right] \\ \Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^*.$$

Proof. For simplicity, from inequality (6) we have that

$$-\varphi(t) \leq u'''(t) + \beta(t)f(u(t))u''(t) \\ + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) - G(t, u(t), u'(t)) \leq \varphi(t).$$

By condition(i) and according to Theorem 2, there exists $\varepsilon \in [t_0, t]$ such that

$$\int_{t_0}^t u'''(s)ds + u''(\varepsilon) \int_{t_0}^t \beta(s)f(u(s))ds + \int_{t_0}^t \alpha(s) \frac{d}{ds} D(u(s))ds \\ + \int_{t_0}^t \rho(s)r(u(s))ds - \int_{t_0}^t G(s, u(s), u'(s)) \leq \int_{t_0}^t \varphi(s)ds. \quad (22)$$

Using integration by parts, since $\alpha(t)$ is nondecreasing, there exists $\tau > 0$ such that $\alpha(t) \geq \tau$, $\alpha'(t) \geq 0$, then inequality (22) becomes

$$\tau |D(u(t))| \leq \int_{t_0}^t \varphi(s)ds + \int_{t_0}^t |u'''(s)|ds + |u''(\varepsilon)| \int_{t_0}^t \beta(s)f(|u(s)|)ds \\ + \int_{t_0}^t \rho(s)r(|u(s)|)ds + \int_{t_0}^t |G(s, u(s), u'(s))|ds. \quad (23)$$

According to conditions (ii) and (iii) of Theorem 3.1 we obtain

$$|D(u(t))| \leq \frac{1}{\tau} \int_{t_0}^t \varphi(s)ds + \frac{\delta}{\tau} + \frac{|u''(\varepsilon)|}{\tau} \int_{t_0}^t \beta(s)f(|u(s)|)ds \\ + \frac{1}{\tau} \int_{t_0}^t \rho(s)r(|u(s)|)ds + \frac{1}{\tau} \int_{t_0}^t \phi(s)\varpi(|u(s)|)(|u'(s)|)^{n+1}ds.$$

From condition(iii) and Theorem 2.1, there exists $\xi \in [t_0, t]$ such that

$$|u(t)| \leq \frac{1+L}{\tau} \int_{t_0}^t \varphi(s)ds + \frac{k_1}{\tau} \int_{t_0}^t \beta(s)f(|u(s)|)ds \\ + \frac{1}{\tau} \int_{t_0}^t \rho(s)r(|u(s)|)ds + \frac{\lambda^{n+1}}{\tau} \int_{t_0}^t \phi(s)\varpi(|u(s)|)ds,$$

where we set $|D(u(t))| \geq |u(t)|$. It is clear from Theorem 2.3 that

$$|u(t)| \leq \frac{(1+\delta)}{\tau} \int_{t_0}^t \varphi(s)ds Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{\tau} \int_{t_0}^t \phi(s)\varpi[\Omega^{-1}(\Omega(1) \\ + \frac{1}{\tau} \int_{t_0}^s \rho(\alpha)r(T(\alpha))d\alpha) T(s)] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{1}{\tau} \int_{t_0}^t \rho(s)r(T(s))ds \right) T(t),$$

where

$$T(t) = F^{-1} \left(F(1) + \frac{k_1}{\tau} \int_{t_0}^t \beta(s)ds \right).$$

We use conditions (iv) and (v) of Theorem 3.1 to get

$$|u(t)| \leq \sigma \frac{(1+\delta)}{\tau} Y^{-1} \left[Y(1) + k_3 \frac{\lambda^{n+1}}{\tau} \varpi \left[\Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^* \right] \right] \\ \Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^*,$$

where

$$T^* = F^{-1} \left(F(1) + k_4 \frac{k_1}{\tau} \right).$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_1} \varphi(t).$$

, where

$$C_{\varphi_1} = \frac{(1+\delta)}{\tau} Y^{-1} \left[Y(1) + k_3 \frac{\lambda^{n+1}}{\tau} \varpi \left[\Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^* \right] \right] \\ \Omega^{-1} \left(\Omega(1) + k_2 \frac{1}{\tau} r(T^*) \right) T^*.$$

Theorem 3.2. Assume that conditions (ii),(iii),(iv),(v) of Theorem 3.1 remain valid. Then, equation (2) has Hyers-Ulam-Rassias stability, where the Hyers-Ulam-Rassias constant(C_{φ_2}) is given as

$$C_{\varphi_2} = \sigma \frac{\lambda}{L} Y^{-1} \left[Y(1) + k_2 \frac{\lambda^{n+1}}{L} \varpi \left[\Omega^{-1} (\Omega(1) + k_3 \frac{\lambda}{L} r(T^*) ds) T^* \right] \right] \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r(T^*) ds \right) T^*,$$

Proof. It is clear from equation (7) that

$$-\varphi(t) \leq u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)r(u(t)) - G(t, u(t), u'(t)) \leq \varphi(t). \quad (24)$$

Then multiplying (24) by $u'(t)$ and consider it in the form

$$u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \rho(t)r(u(t))u'(t) - G(t, u(t), u'(t))u'(t) \\ \leq u'(t)\varphi(t).$$

For simplicity, we use equation (5) and theorem 2.1, there exists $\eta, \mu, \xi, \tau \in [t_0, t]$ such that

$$u'''(\xi)u(t) \leq u'(\eta) \int_{t_0}^t \varphi(s)ds - u''(\mu)u'(\mu) \int_{t_0}^t \beta(s)f(u(s))ds \\ - u'(\tau) \int_{t_0}^t \rho(s)r(u(s))ds + \int_{t_0}^t G(s, u(s), u'(s))u'(s)ds.$$

Then, by conditions (ii),(iii) of Theorem 3.1 and Theorem 2.1, there exists $\varepsilon \in [t_0, t]$ such that

$$|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds + \frac{\lambda k_1}{L} \int_{t_0}^t \beta(s)f(|u(s)|)ds \\ + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(|u(s)|)ds + \frac{\lambda^{n+1}}{L} \int_{t_0}^t \phi(t)\varpi(|u(t)|)ds.$$

We apply Theorem 2.3 to obtain

$$|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{L} \int_{t_0}^t \phi(s)\varpi[\Omega^{-1}(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^s \rho(\alpha)r(T(\alpha))d\alpha) T(s)] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(T(s))ds \right) T(t),$$

where

$$T(t) = F^{-1} \left(F(1) + \frac{k_1}{L} \int_{t_0}^t \beta(s)ds \right)$$

Using conditions (iv),(v) of Theorem 3.1

$$|u(t)| \leq \sigma \frac{\lambda}{L} \varphi(t) Y^{-1} \left[Y(1) + k_2 \frac{\lambda^{n+1}}{L} \varpi \left[\Omega^{-1} (\Omega(1) + k_3 \frac{\lambda}{L} r(T^*) ds) T^* \right] \right] \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r(T^*) ds \right) T^*,$$

Where:

$$T^* = F^{-1} \left(F(1) + \frac{\lambda k_1}{L} k_4 \right).$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_2} \varphi(t),$$

Where:

$$C_{\varphi_2} = \sigma \frac{\lambda}{L} Y^{-1} \left[Y(1) + k_2 \frac{\lambda^{n+1}}{L} \varpi [\Omega^{-1}(\Omega(1) + k_3 \frac{\lambda}{L} r(T^*)) T^*] \right] \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r(T^*) ds \right) T^*,$$

Theorem 3.3 Assume that conditions (ii),(iii),(iv),(v) of Theorem 3.1 hold. Then, equation (3) has Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_3} = \sigma \frac{\lambda}{L} Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{L} k_2 \varpi [\Omega^{-1}(\Omega(1) + \frac{\lambda}{L} k_3 r(T^*)) T^*] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T^*) ds \right) T^*,$$

Proof. Evaluating inequality (7), multiplying by $u'(t)$ and integrating from t_0 to t to obtain

$$\int_{t_0}^t u'''(s)u'(s)ds + \int_{t_0}^t \alpha(s)g(u(s))(u'(s))ds + \int_{t_0}^t \rho(s)r(u(s))u'(s)ds - \int_{t_0}^t G(s, u(s), u'(s))u'(s)ds \leq \int_{t_0}^s u'(s)\varphi(s)ds. \quad (25)$$

We use equation (5) and Theorem 2, If there exists $\psi, \iota, \xi, \eta \in [t_0, t]$ such that

$$u'''(\psi)u(t) \leq u'(\eta) \int_{t_0}^s \varphi(s)ds - u'(\iota) \int_{t_0}^t \alpha(s)g(u(s))ds - u'(\xi) \int_{t_0}^t \rho(s)r(u(s))ds + \int_{t_0}^t G(s, u(s), u'(s))u'(s)ds.$$

Taking the absolute value of both sides, using conditions (ii) and (iii) of Theorem 3.1 and Theorem 2.1, there exists $v \in [t_0, t]$ such that

$$L|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds + \frac{\lambda}{L} \int_{t_0}^t \alpha(s)g(|u(s)|)ds + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(|u(s)|)ds + \frac{\lambda^{n+1}}{L} \int_{t_0}^t \phi(t)\varpi(|u(t)|)ds.$$

By applying Theorem 2.3, we get

$$|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{L} \int_{t_0}^t \phi(s)\varpi[\Omega^{-1}(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^s \rho(\eta)r(T(\eta))d\eta) T(s)] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(T(s))ds \right) T(t),$$

where

$$T(t) = F^{-1} \left(F(1) + \frac{\lambda}{L} \int_{t_0}^t \alpha(s)ds \right)$$

Using conditions (iv) and (v) of Theorem 3.2, we have

$$|u(t)| \leq \sigma \frac{\lambda}{L} \varphi(t) Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{L} k_2 \varpi [\Omega^{-1}(\Omega(1) + \frac{\lambda}{L} k_3 r(T^*)) T^*] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T^*) ds \right) T^*,$$

where

$$T^* = F^{-1} \left(F(1) + \frac{k_5 \lambda}{L} \right)$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_3} \varphi(t),$$

where

$$C_{\varphi_3} = \sigma \frac{\lambda}{L} Y^{-1} \left[Y(1) + \frac{\lambda^{n+1}}{L} k_2 \varpi [\Omega^{-1}(\Omega(1) + \frac{\lambda}{L} k_3 r(T^*)) T^*] \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T^*) ds \right) T^*,$$

Theorem 3.4. Suppose that the conditions (iii), (iv), (v) of Theorem 3.1 are satisfied, $\varphi: \mathbf{I} \rightarrow (0, \infty)$ a positive, continuous function. In addition, let

1. $Q(t, u(t)) = v(t)\alpha(t)$ where $v(t)$ a continuous function on \mathbf{R}_+ ,
2. $H(t, u(t), u'(t)) = \phi(t)g(u(t))h(u'(t))^4$
3. $\lim_{t \rightarrow \infty} \int_{t_0}^t q(s)ds \leq k_6$, where $k_6 > 0$, $\lim_{t \rightarrow \infty} \int_{t_0}^t p(s)v(s)\alpha(s)ds \leq k_7$,

where $k_7 > 0$,

hold, for function $\alpha(u(t)) \in \Psi$ continuous, nondecreasing and monotonic, then equation (4) is Hyers-Ulam-Rassias stable with initial conditions (5).

Proof. For simplicity, inequality (9) is multiplied by $u'(t)$ to have

$$\begin{aligned} -u'(t)\varphi(t) &\leq u'''(t)u'(t) + np(t)Q(t, u(t))(u'(t))u'(t) \\ &+ q(t)f(u(t))u'(t) - H(t, u(t), u'(t))u'(t) \leq u'(t)\varphi(t). \end{aligned} \quad (26)$$

We use conditions (i'), (ii') of Theorem 3.4, integrate from t_0 to t and use Theorem 2.1, there exist $\kappa, \iota, \rho, \varepsilon \in [t_0, t]$ such that

$$\begin{aligned} u'''(\kappa) \int_{t_0}^t u'(s)ds + nu'(\xi)^2 \int_{t_0}^t p(s)v(s)\alpha(s)ds \\ + u'(\iota) \int_{t_0}^t q(s)f(u(s))ds - h(u'(\rho))^4 u'(\rho) \int_{t_0}^t \phi(s)g(u(s))ds \leq u'(\varepsilon) \int_{t_0}^t \varphi(s)ds. \end{aligned}$$

Using equation (5) and condition (iii) of Theorem 3.1 to obtain

$$\begin{aligned} |u(t)| &\leq \left(\frac{\lambda}{L} + \frac{n\lambda}{L} \int_{t_0}^t p(s)v(s)\alpha(s)ds \right) \int_{t_0}^t \varphi(s)ds \\ &+ \frac{\lambda}{L} \int_{t_0}^t q(s)f(u(s))ds + \frac{\lambda(h(\lambda))^4}{L} \int_{t_0}^t \phi(s)g(u(s))ds. \end{aligned}$$

According to Corollary 1 we have

$$\begin{aligned} |u(t)| &\leq \left(\frac{\lambda}{L} + \frac{n\lambda}{L} \int_{t_0}^t p(s)v(s)\alpha(s)ds \right) \int_{t_0}^t \varphi(s)ds \\ &\Omega^{-1} \left(\Omega(1) + \frac{\lambda(h(\lambda))^4}{L} \int_{t_0}^t \phi(s)g \left(F^{-1} \left(F(1) + \frac{n\lambda}{L} \int_{t_0}^t q(\alpha)d\alpha \right) \right) ds \right) \\ &F^{-1} \left(F(1) + \frac{n\lambda}{L} \int_{t_0}^t q(s)ds \right), \quad t \in \mathbf{I}. \end{aligned}$$

Applying conditions (iv), (v) of Theorem 3.1 and (iii') of Theorem 3.4, we arrive at

$$\begin{aligned} |u(t)| &\leq \sigma \left(\frac{\lambda}{L} + k_7 \frac{n\lambda}{L} \right) \varphi(t) \\ &\Omega^{-1} \left(\Omega(1) + k_2 \frac{\lambda(h(\lambda))^4}{L} g \left(F^{-1} \left(F(1) + k_6 \frac{n\lambda}{L} \int_{t_0}^t q(\alpha)d\alpha \right) \right) ds \right) \\ &F^{-1} \left(F(1) + k_6 \frac{n\lambda}{L} \right), \quad t \in \mathbf{I}. \end{aligned}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_{\varphi} \varphi(t),$$

where

$$\begin{aligned} C_{\varphi_4} &= \sigma \left(\frac{\lambda}{L} + k_7 \frac{n\lambda}{L} \right) \\ \Omega^{-1} \left(\Omega(1) + k_2 \frac{\lambda(h(\lambda))^4}{L} g \left(F^{-1} \left(F(1) + k_6 \frac{n\lambda}{L} \int_{t_0}^t q(\alpha) d\alpha \right) \right) ds \right) \\ F^{-1} \left(F(1) + k_6 \frac{n\lambda}{L} \right). \end{aligned}$$

If we put $G(t, u(t), u'(t)) = 0$ in equations (1),(2),(3) and $H(t, u(t), u'(t)) = 0$ in (4) the results are given by the following theorems:

Theorem 3.5: The nonlinear differential equation

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) = 0 \quad (27)$$

has Hyers-Ulam-Rassias stability property with initial conditions (5) with conditions (iii), (iv), (v) of Theorem 3.1 remained valid. Then, the Hyers-Ulam-Rassias constant is given as

$$\begin{aligned} C_{\varphi_5} \sigma \frac{\lambda}{L} \Upsilon^{-1} \left[\Upsilon(1) + \frac{k_2 \lambda^{n+1}}{L} \varpi[\Omega^{-1}(\Omega(1) \right. \\ \left. + \frac{k_3 \lambda}{L} r(T*)) T* \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T*) \right) T*, \end{aligned}$$

Proof. We consider inequality (4) as

$$\begin{aligned} \int_{t_0}^t u'''(s)u'(s)ds + \int_{t_0}^t \beta(s)f(u(s))u''(s)u'(s)ds + \int_{t_0}^t \alpha(s)g(u(s))(u'(s))^2 ds \\ + \int_{t_0}^t \rho(s)r(u(s))u'(s)ds \leq \int_{t_0}^t u'(s)\varphi(s). \end{aligned}$$

It is clear by equation (5) and Theorem 2.1, there exists $\varepsilon, \kappa, \xi, \eta, \gamma, \rho \in [t_0, t]$ such that

$$\begin{aligned} u'''(\varepsilon)u(t) \leq u'(\gamma) \int_{t_0}^t \varphi(s)ds - u''(\kappa)u'(\kappa) \int_{t_0}^t \beta(s)f(u(s))ds - (u'(\xi))^2 \int_{t_0}^t \alpha(s)g(u(s))ds \\ - u'(\eta) \int_{t_0}^t \rho(s)r(u(s))ds. \end{aligned}$$

Taking the absolute value and applying conditions (iii) of Theorem 3.1 to have

$$\begin{aligned} u(t) \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds + \frac{\lambda k_1}{L} \int_{t_0}^t \beta(s)f(|u(s)|)ds + \frac{\lambda^2}{L} \int_{t_0}^t \alpha(s)g(|u(s)|)ds \\ + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(|u(s)|)ds. \end{aligned}$$

By applying Theorem 2.3, we get

$$\begin{aligned} |u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds \Upsilon^{-1} \left[\Upsilon(1) + \frac{\lambda^{n+1}}{L} \int_{t_0}^t \phi(s) \varpi[\Omega^{-1}(\Omega(1) \right. \\ \left. + \frac{\lambda}{L} \int_{t_0}^s \rho(\eta)r(T(\eta))d\eta) T(s) \right] ds \Omega^{-1} \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(T(s))ds \right) T(t), \end{aligned}$$

where

$$T(t) = F^{-1} \left(F(1) + \frac{\lambda}{L} \int_{t_0}^t \alpha(s)ds \right)$$

Using conditions(iv) and (v) of Theorem 3 we have

$$\begin{aligned} |u(t)| \leq \sigma \frac{\lambda}{L} \varphi(t) \Upsilon^{-1} \left[\Upsilon(1) + \frac{k_2 \lambda^{n+1}}{L} \varpi[\Omega^{-1}(\Omega(1) \right. \\ \left. + \frac{k_3 \lambda}{L} r(T*)) T* \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T*) \right) T*, \end{aligned}$$

where

$$T * = F^{-1} \left(F(1) + \frac{k_5 \lambda}{L} \right)$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_4} \varphi(t),$$

where

$$C_{\varphi_5} \sigma \frac{\lambda}{L} Y^{-1} \left[Y(1) + \frac{k_2 \lambda^{n+1}}{L} \varpi [\Omega^{-1}(\Omega(1) + \frac{k_3 \lambda}{L} r(T *))] T * \right] \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r(T *) \right) T *,$$

Theorem 3.6. Assume that conditions (ii),(iii),(iv),(v) of Theorem 3.1 remain valid. Then, equation

$$u'''(t) + \beta(t)f(u(t))u''(t) + \rho(t)r(u(t)) = 0 \quad (28)$$

has Hyers-Ulam-Rassias stability, where the Hyers-Ulam-Rassias constant(C_{φ}) is given as

$$C_{\varphi_6} = \sigma \left(\frac{\sigma \lambda}{L} \right) \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_4 k_1 \lambda}{L} \right) \right) \right) F^{-1} \left(F(1) + \frac{k_4 k_1 \lambda}{L} \right)$$

Proof. It is clear from equation (7) as we take $G(t, u(t), u'(t)) = 0$ to have

$$u'''(t)u'(t) + \beta(t)f(u(t))u''(t)u'(t) + \rho(t)r(u(t))u'(t) \leq u'(t)\varphi(t).$$

Integrating and using Theorem 2, there exists $\eta, \mu, \xi, \tau \in [t_0, t]$ such that

$$u'''(\xi) \int_{t_0}^t u'(s)ds + u''(\mu)u'(\mu) \int_{t_0}^t \beta(s)f(u(s))ds + u'(\tau) \int_{t_0}^t \rho(s)r(u(s))ds \leq u'(\eta) \int_{t_0}^t \varphi(s)ds.$$

Then, using equation (5), conditions (ii),(iii) of Theorem 3.1 and Theorem 2.1, there exists $\varepsilon \in [t_0, t]$ such that

$$|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds + \frac{\lambda k_1}{L} \int_{t_0}^t \beta(s)f(|u(s)|)ds + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(|u(s)|)ds$$

We apply Corollary1 to obtain

$$|u(t)| \leq \left(\frac{\lambda}{L} \right) \int_{t_0}^t \varphi(s)ds + \Omega^{-1} \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r \left(F^{-1} \left(F(1) + \frac{k_1 \lambda}{L} \int_{t_0}^t \beta(\alpha)d\alpha \right) \right) ds \right) F^{-1} \left(F(1) + \frac{k_1 \lambda}{L} \int_{t_0}^t \beta(s)ds \right), \quad t \in \mathbf{I},$$

According to conditions (iv),(v) of Theorem 3.1 we obtain

$$|u(t)| \leq \sigma \left(\frac{\sigma \lambda}{L} \right) \varphi(t) \Omega^{-1} \left(\Omega(1) + \frac{k_3 \lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_4 k_1 \lambda}{L} \right) \right) \right) F^{-1} \left(F(1) + \frac{k_4 k_1 \lambda}{L} \right), \quad t \in \mathbf{I},$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_5} \varphi(t),$$

where

$$C_{\varphi_6} = \sigma\left(\frac{\sigma\lambda}{L}\right) \Omega^{-1} \left(\Omega(1) + \frac{k_3\lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_4k_1\lambda}{L} \right) \right) \right) \\ F^{-1} \left(F(1) + \frac{k_4k_1\lambda}{L} \right)$$

Theorem 3.7. Assume that conditions (ii),(iv),(v) of Theorem 3.1 hold. Then, equation $u'''(t) + \alpha(t)g(u(t))u'(t) + \rho(t)r(u(t)) = 0$ is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant defined as

$$C_{\varphi_7} = \sigma\left(\frac{\lambda}{L}\right) \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right) \right) \right) \\ F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right)$$

Proof. From inequality (8), it is clear that

$$\int_{t_0}^t u'''(s)u'(s)ds + \int_{t_0}^t \alpha(s)g(u(s))(u'(s))ds + \int_{t_0}^t \rho(s)r(u(s))u'(s)ds \\ \leq \int_{t_0}^s u'(s)\varphi(s)ds. \quad (30)$$

By Theorem 2, If there exists $\psi, \iota, \eta \in [t_0, t]$ such that

$$u'''(\psi) \int_{t_0}^t u'(s)ds + u'(\iota) \int_{t_0}^t \alpha(s)g(u(s))ds + u'(\xi) \int_{t_0}^t \rho(s)r(u(s))ds \\ \leq u'(\eta) \int_{t_0}^s \varphi(s)ds.$$

According to equation (5) and condition (ii) of Theorem 3.1 we get

$$|u(t)| \leq \frac{\lambda}{L} \int_{t_0}^t \varphi(s)ds + \frac{\lambda}{L} \int_{t_0}^t \alpha(s)g(|u(s)|)ds + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r(|u(s)|)ds$$

By applying Corollary 1, we get

$$|u(t)| \leq \left(\frac{\lambda}{L}\right) \int_{t_0}^t \varphi(s)ds \\ \Omega^{-1} \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t \rho(s)r \left(F^{-1} \left(F(1) + \frac{\lambda}{L} \int_{t_0}^t \alpha(\iota)d\iota \right) \right) ds \right) \\ F^{-1} \left(F(1) + \frac{\lambda}{L} \int_{t_0}^t \alpha(s)ds \right), \quad t \in \mathbf{I},$$

According to conditions (iv) and (v) of Theorem 3.1 we obtain

$$|u(t)| \leq \sigma\left(\frac{\lambda}{L}\right) \varphi(t) \\ \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right) \right) ds \right) \\ F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right), \quad t \in \mathbf{I},$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_3} \varphi(t),$$

where

$$C_{\varphi_7} = \sigma\left(\frac{\lambda}{L}\right) \Omega^{-1} \left(\Omega(1) + k_3 \frac{\lambda}{L} r \left(F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right) \right) \right) \\ F^{-1} \left(F(1) + \frac{k_5\lambda}{L} \right)$$

Theorem 3.8 Let the conditions (iv),(v) of Theorem 3.1 and the conditions (i'),(iii') of the Theorem

3.3 hold. Then, the equation

$$u'''(t) + np(t)Q(t, u(t))(u'(t)) + q(t)f(u(t)) = 0, \quad (31)$$

has the Hyers-Ulam-Rassias stability where Hyers-Ulam-Rassias constant is given as

$$C_{\varphi_8} = \sigma \left(\frac{\lambda}{L} + \frac{nk_7\lambda}{L} \right) \Omega^{-1} \left(\Omega(1) + \frac{k_5\lambda}{L} \right)$$

Proof. If $H(t, u(t), u'(t)) = 0$ then, inequality (8) is multiplied by $u'(t)$ to get

$$\begin{aligned} & \int_{t_0}^t u'(s)u'''(s)ds + n \int_{t_0}^t p(s)Q(s, u(s))(u'(s))^2 ds \\ & + \int_{t_0}^t q(s)f(u(s))u'(s)ds \leq \int_{t_0}^t u'(s)\varphi(s)ds. \end{aligned}$$

We use condition(i') of Theorem 3.4 and apply Theorem 2.1, there exists $\xi, \eta, \kappa, \varepsilon \in [t_0, t]$ such that

$$\begin{aligned} & u'''(\xi) \int_{t_0}^t u'(s)ds + n(u'(\eta))^2 \int_{t_0}^t p(s)v(s)\alpha(s)ds \\ & + u'(\kappa) \int_{t_0}^t q(s)f(u(s))ds \leq u'(\varepsilon) \int_{t_0}^t \varphi(s)ds. \end{aligned}$$

Using condition (iv) of Theorem 3.1, equation (5) we obtain

$$\begin{aligned} |u(t)| & \leq \left(\frac{\lambda}{L} + \frac{n\lambda}{L} \int_{t_0}^t p(s)v(s)\alpha(s)ds \right) \int_{t_0}^t \varphi(s)ds \\ & + \frac{\lambda}{L} \int_{t_0}^t q(s)f(u(s))ds. \end{aligned}$$

According to Theorem 2.2 we have

$$|u(t)| \leq \left(\frac{\lambda}{L} + \frac{n\lambda}{L} \int_{t_0}^t p(s)v(s)\alpha(s)ds \right) \int_{t_0}^t \varphi(s)ds \Omega - 1 \left(\Omega(1) + \frac{\lambda}{L} \int_{t_0}^t q(s)ds \right), \quad t \in \mathbf{I},$$

Using conditions (v) of Theorem 3 and condition(iii'') of Theorem 5(')@ to arrive at

$$|u(t)| \leq \sigma \left(\frac{\lambda}{L} + \frac{nk_7\lambda}{L} \right) \varphi(t) \Omega^{-1} \left(\Omega(1) + \frac{k_5\lambda}{L} \right), \quad t \in \mathbf{I}.$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_{\varphi_6} \varphi(t),$$

where

$$C_{\varphi_8} = \sigma \left(\frac{\lambda}{L} + \frac{nk_7\lambda}{L} \right) \Omega^{-1} \left(\Omega(1) + \frac{k_5\lambda}{L} \right).$$

Illustrative Examples

Example 1. Consider the Hyers-Ulam-Rassias stability of the third order nonlinear differential equation of the form

$$u'''(t) + t^{-6}u^4(t)u''(t) + t^{-4}u(t) = t^{-4}u'(t)u'^4(t)u^2(t), \quad t > 0$$

where

$$G(t, u(t), u'(t)) = t^{-4}u'(t)^4u^2(t)$$

with the initial condition (3). By conditions of Theorem 3.1 that satisfied the Theorem 3.4 third order nonlinear differential equation is Hyers-Ulam-Rassias stable.

Example 2. Consider the Hyers-Ulm-Rassias stability of third order nonlinear differential equation of the form

$$u'''(t) + t^{-2}u^2(t)u''(t) + t^{-2}u(t)u'(t) + t^{-3}u^4(t) = 0, \quad t > 0,$$

with the initial conditions (5). By the conditions of Theorem 3.1, the third order nonlinear differential equation is Hyers-Ulam-Rassias stable.

Example 3. Consider the Hyers-Ulm-Rassias stability of equation

$$u'''(t) + t^{-6}u^2(t)u''(t) + t^{-2}u(t)u'(t) + t^{-3}u^4(t) = t^{-4}u'^4u^2(t) \quad t > 0$$

where

$$G(t, u(t), u'(t)) = t^{-4}u'^4u^2(t),$$

$\beta(t) = t^{-6}$, $\alpha(t) = t^{-4}$, $\rho(t) = t^{-3}$ with the initial condition (5). By conditions of Theorem 3.1, third order nonlinear differential equation is Hyers-Ulam-Rassias stable.

IV. CONCLUSION

This work on Hyers-Ulam-Rassias stability results of third order nonlinear differential equations is very prominent in the stability of some problems such as hereditary, the surge in birth-rates, spreading of certain contagious diseases and so on. These problems appear directly in terms of integral equations and in terms of nonlinear differential equations with certain conditions on the parameters of the model which can assist us to reduce the nonlinear equation to integral equations.

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