

## Chebyshev Polynomials and Their Applications in Numerical Analysis

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**ABSTRACT** Differential equations play a crucial role in modeling diverse phenomena in science and technology, including growth, decay, epidemics, and oscillatory systems. Number-based methods, especially for solving first-order ordinary differential equations (ODEs), have grown because people want quick and correct answers to these equations. The main goal of this study is to come up with and evaluate a numerical method for using Chebyshev polynomials of the first kind to solve general first-order ODEs. Chebyshev polynomials are famous for being orthogonal and having uses in approximation theory. They are also used to build continuous linear multistep methods (LMMs). These methods utilize interpolation and collocation techniques to achieve high accuracy. The Chebyshev method is systematically derived and analyzed for order, consistency, zero-stability and convergence. Employing boundary locus methods, the stability region is determined, affirming the method's robustness. The numerical applications include solving problems such as the Malthusian growth model, Prothero-Robinson oscillatory equations and highly stiff differential equations. Comparisons of absolute errors with existing methods highlight the Chebyshev method's superior accuracy and efficiency. Results show minimal errors across various test cases, confirming its applicability to real-world problems involving oscillatory and stiff systems. This study contributes to advancing numerical analysis by providing a reliable, efficient tool for solving initial value problems. The findings emphasize the significance of Chebyshev polynomials in enhancing the accuracy and stability of numerical schemes, offering potential for broader applications in science and engineering disciplines.

**KEYWORDS:** Chebyshev polynomials, Numerical methods, Linear multistep methods, Oscillatory models, Stiff equations.

### I. INTRODUCTION

Chebyshev polynomials stand out in numerical analysis due to their unique ability to address challenges that traditional methods struggle to overcome, particularly in systems exhibiting oscillatory or stiff behavior. Unlike conventional polynomial techniques, Chebyshev-based methods are designed to minimize the Runge phenomenon, an issue characterized by oscillatory errors in polynomial interpolation at equidistant points (Ruiyi et al. 2021). By employing

Chebyshev nodes, which are non-uniformly distributed points optimized for accuracy, these methods ensure superior stability and error reduction, even in complex approximation problems. This feature makes them a novel and highly efficient tool in applications requiring precision, such as fluid dynamics, electromagnetic simulations, and quantum mechanics (Trefethen, 2020).

What sets Chebyshev-based methods apart is their exceptional suitability for solving oscillatory and stiff systems. These systems are notorious for their rapid variations or drastic changes in behavior over small intervals, which can destabilize traditional numerical techniques. Chebyshev polynomials mitigate these challenges through their orthogonality and spectral convergence properties (Donatella and Woula, 2021). These characteristics enable the efficient representation of oscillatory solutions and allow the accurate resolution of stiff problems without excessive computational costs (Xiaolong, 2021). Recent advancements in numerical analysis have demonstrated that Chebyshev spectral methods outperform traditional finite difference and finite element methods in handling such systems, offering faster convergence rates and reduced computational complexity (Boyd, 2021; Xiaolong, 2023).

Moreover, the novelty of Chebyshev-based methods lies in their adaptability to modern computational challenges. In areas like machine learning, signal processing, and high-performance computing, where large-scale, oscillatory, or stiff problems are prevalent, these methods provide unparalleled efficiency (Xiaolong and John, 2023). They excel in scenarios requiring real-time computation or large data sets, as their mathematical foundation ensures robustness against numerical instability (Xiang, 2018). This adaptability, combined with their ability to integrate seamlessly with hybrid numerical techniques, cements Chebyshev-based methods as a transformative approach in numerical analysis, offering a clear advantage over existing techniques in terms of precision, stability, and computational efficiency (Chien-Cheng and Su-Ling, 2021; Canuto et al., 2022).

In recent times, classic applications of differential equations have been found in many areas of science and technology. They can be used for modeling of physical, technical, or biological processes such as in the study of growth, decay, epidemics, electricity, among others. The main questions of modern technology are how to increase the accuracy of calculations considering short computational time and how to decrease necessary mathematical operations (Aboiyar et al., 2015). In a bid to model real-life problems in areas of engineering, biological sciences, physical sciences, electronics, and many others, initial value problems are most times encountered (Adeniyi and Alabi, 2017; Shakeri and Dehghan, 2018).

In this study we shall be concerned with the construction and analysis of an efficient numerical method for the approximate solution of the general first order differential equations of the form:

$$y' = f(x, y), y(a) = \eta, f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad (1.1)$$

It is general knowledge that differential equations model quite a lot of physical problems that occur in many fields of sciences, mathematics, and engineering; common among such differential equations are ordinary differential equations. Thus, the solutions to these differential equations and consequently the physical problems that they represent a subject of interest to mathematicians, particularly Numerical Analysts. A wide range of numerical schemes for solving initial value problems of first-order ordinary differential equations using different approaches have been developed and are continuously being sought, with extension to higher-order ordinary differential equations. Listing just a few of such research would include Ayinde and Ibijola (2015), Taiwo (2015), Aboiyar et al. (2015), Olanegan et al. (2016), Fadugba and Falodun (2017), Fadugba and Okunlola (2017), Aksah et al. (2019), Fadugba et al. (2020), among others numerous

numerical integrators for the solution of initial value problems of various orders. Many of the above-mentioned methods employed interpolation and collocation approach in deriving schemes of varying accuracy for first order initial value problems (1.1).

Several continuous linear multistep methods (LMMs) have been derived using interpolation and collocation points for the solution of (1.1) with constant step size using Chebyshev polynomials as a basis. Chebyshev polynomials are a well-known family of orthogonal polynomials on the interval  $[-1, 1]$  associated with certain weight function. Chebyshev polynomials are of great importance in many areas of mathematics, particularly in approximation theory. There are several kinds of Chebyshev polynomials: Adegboro and Kayode (2018), Panopoulos et al. (2023), Kamoh et al. (2018), Ndanusa and Tafida (2016), Adeniyi and Alabi (2017). The most important kinds of the Chebyshev polynomials is Chebyshev polynomials  $T_n(x)$  of first kind. The Chebyshev polynomial  $T_n(x)$  of the first kind is a polynomial in  $x$  of degree  $n$ , defined by the relation by Vlastic et al. (2019) as follows

$$T_n(x) = \cos(n \arccos x), \text{ where } n \geq 0 \text{ and } x \in [-1, 1] \quad (1.2)$$

There are several ways to develop the Chebyshev polynomials, but there seems to be nothing in the literature that constructs the Chebyshev polynomials of the first kind by explicitly constructing their coefficients. Two methods for explicitly constructing the coefficients of the Chebyshev polynomial equations of the first kind are presented here. Both of these methods begin with the recurrence relation,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (1.3)$$

and with initial conditions

$$T_0(x) = 1 \text{ and } T_1(x) = x \quad (1.4)$$

where  $T_n(x)$  denotes the Chebyshev polynomials of the first kind. Both also compute each  $T_n^k(0)$  for use in the Taylor series expansion,

$$T_n(x) = T_n(0) + (x-0) \frac{T_n'(0)}{1!} + (x-0)^2 \frac{T_n''(0)}{2!} + \dots + (x-0)^k \frac{T_n^k(0)}{k!} \quad (1.5)$$

where  $T_n^k(0)$  denotes the  $k^{th}$  derivative of the  $n^{th}$  Chebyshev polynomial evaluated at zero.

### Theorem 1.1

**Roots of Chebyshev Polynomials:** The roots of  $T_n(x)$  of degree  $n \leq 1$  has  $n$  simple zeros in  $[-1, 1]$  at  $\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right)$  for each  $k = 1, 2, \dots, n$ . Moreover,  $T_n(x)$  assumes its absolute extrema at

$$\bar{x}_k^j = \cos\left(\frac{2k}{n}\right), \text{ with } T_n\left(\bar{x}_k^j\right) = (-1)^k, \text{ for each } k = 0, 1, \dots, n.$$

The Chebyshev polynomial possess the following properties.

$$\text{i. } |T_n(x)| \leq 1, x \in [-1, 1]$$

- ii.  $T_n(x)$  is a polynomial of degree  $n$ . If  $n$  is even  $T_n(x)$  is an even polynomial and if  $n$  is odd,  $T_n(x)$  is an odd polynomial.
- iii.  $T_n(x)$  assumes extreme value at  $n+1$  points  $x_k = \cos(k\pi/n)$ ,  $k=0, 1, \dots, n$  and extreme value at  $x_k$  is  $(-1)^k$ .
- iv.  $T_n(x)$  is orthogonal with respect to the weight function  $w(x) = \frac{1}{\sqrt{1-x^2}}$  and

$$\int \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}} dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}.$$

Chebyshev polynomials are a sequence of orthogonal polynomials that arise in the context of approximating functions and solving differential equations. There are two kinds, Chebyshev polynomials of the first kind  $T_n(x)$  and the second kind,  $U_n(x)$ , defined over the interval  $[-1, 1]$ . The roots of Chebyshev polynomials are also used as nodes in Gaussian quadrature, contributing to their widespread application in numerical methods.

The Chebyshev polynomial of the first kind  $T_n(x)$  is a polynomial of degree  $n$  defined for  $x \in [-1, 1]$  by

$$T_n(x) = \cos(n \arccos x), \quad -1 \leq T_n \leq 1 \quad (1.6)$$

By setting  $x = \cos z$ , we have

$$T_n = \cos nz \quad (1.7)$$

from which it is easy to deduce the expressions for the recursive relation of Chebyshev polynomials as,  $T_0 = 1, T_1 = \cos z, T_2 = \cos 2z = 2\cos^2 z - 1 = 2x^2 - 1, \dots, T_{n+1} = 2xT_n - T_{n-1}, n \leq 1$ .

The  $T_n$  is a polynomial of degree  $n$  with leading coefficient  $2^{n-1}$  for  $n \leq 1$ .

## II. DERIVATION OF CHEBYSHEVIAN METHOD

We consider the first six terms of the basis function (1.7) as our approximate solution. This is given by

$$y(x) = T_0(x) + T_1(x) + \sum_{n=1}^1 (2xT_n(x) - T_{n-1}(x)) \quad (2.1)$$

We seek the solution of (1.1) on the partition  $\pi_N = x_0 < x_1 < x_2 < \dots < x_n < x_{n-1} < \dots < \pi_N = b$  of the integration interval  $[a, b]$  with a constant step-size  $h$ , given by,  $h = x_{n+1} - x_n, n = 0, 1, 2, \dots, N$ .

Now interpolating (2.1) at point  $x_{n+s}, s = 0$  and collocating its first derivative at points

$x_{n+r}, r = 0\left(\frac{1}{7}\right)1$ , leads to the following system of equations,

$$XA = U \quad (2.2)$$

Where

$$A = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \ a_8]^T, \quad U = \left[ f_n \ f_{n+\frac{1}{7}} \ f_{n+\frac{2}{7}} \ f_{n+\frac{3}{7}} \ f_{n+\frac{4}{7}} \ f_{n+\frac{5}{7}} \ f_{n+\frac{6}{7}} \ f_{n+1} \right]^T \text{ and}$$

$$X = \begin{bmatrix} 1 & -4x_n & -20x_n^2 & 40x_n^3 & 120x_n^4 & -96x_n^5 & -224x_n^6 & 68x_n^7 & 128x_n^8 \\ 0 & -4 & -40x_n & 120x_n^2 & 480x_n^3 & 480x_n^4 & -1344x_n^5 & 448x_n^6 & 1024x_n^7 \\ 0 & -4 & -40x_{n+\frac{1}{7}} & 120x_{n+\frac{1}{7}}^2 & 480x_{n+\frac{1}{7}}^3 & 480x_{n+\frac{1}{7}}^4 & -1344x_{n+\frac{1}{7}}^5 & 448x_{n+\frac{1}{7}}^6 & 1024x_{n+\frac{1}{7}}^7 \\ 0 & -4 & -40x_{n+\frac{2}{7}} & 120x_{n+\frac{2}{7}}^2 & 480x_{n+\frac{2}{7}}^3 & 480x_{n+\frac{2}{7}}^4 & -1344x_{n+\frac{2}{7}}^5 & 448x_{n+\frac{2}{7}}^6 & 1024x_{n+\frac{2}{7}}^7 \\ 0 & -4 & -40x_{n+\frac{3}{7}} & 120x_{n+\frac{3}{7}}^2 & 480x_{n+\frac{3}{7}}^3 & 480x_{n+\frac{3}{7}}^4 & -1344x_{n+\frac{3}{7}}^5 & 448x_{n+\frac{3}{7}}^6 & 1024x_{n+\frac{3}{7}}^7 \\ 0 & -4 & -40x_{n+\frac{4}{7}} & 120x_{n+\frac{4}{7}}^2 & 480x_{n+\frac{4}{7}}^3 & 480x_{n+\frac{4}{7}}^4 & -1344x_{n+\frac{4}{7}}^5 & 448x_{n+\frac{4}{7}}^6 & 1024x_{n+\frac{4}{7}}^7 \\ 0 & -4 & -40x_{n+\frac{5}{7}} & 120x_{n+\frac{5}{7}}^2 & 480x_{n+\frac{5}{7}}^3 & 480x_{n+\frac{5}{7}}^4 & -1344x_{n+\frac{5}{7}}^5 & 448x_{n+\frac{5}{7}}^6 & 1024x_{n+\frac{5}{7}}^7 \\ 0 & -4 & -40x_{n+\frac{6}{7}} & 120x_{n+\frac{6}{7}}^2 & 480x_{n+\frac{6}{7}}^3 & 480x_{n+\frac{6}{7}}^4 & -1344x_{n+\frac{6}{7}}^5 & 448x_{n+\frac{6}{7}}^6 & 1024x_{n+\frac{6}{7}}^7 \\ 0 & -4 & -40x_{n+1} & 120x_{n+1}^2 & 480x_{n+1}^3 & 480x_{n+1}^4 & -1344x_{n+1}^5 & 448x_{n+1}^6 & 1024x_{n+1}^7 \end{bmatrix}$$

Solving (2.2), for  $0 \leq t \leq h$  and substituting back into (2.1) gives a continuous linear multistep method of the form,

$$y(x) = \alpha_0(x)y_n + h \left[ \beta_0(x)f_n + \beta_{\frac{1}{7}}(x)f_{n+\frac{1}{7}} + \beta_{\frac{2}{7}}(x)f_{n+\frac{2}{7}} + \beta_{\frac{3}{7}}(x)f_{n+\frac{3}{7}} + \beta_{\frac{4}{7}}(x)f_{n+\frac{4}{7}} + \beta_{\frac{5}{7}}(x)f_{n+\frac{5}{7}} + \beta_{\frac{6}{7}}(x)f_{n+\frac{6}{7}} + \beta_1(x)f_{n+1} \right] \quad (2.3)$$

where

$$\alpha_0 = 1$$

$$\beta_0 = t h - \frac{363}{40} t^2 h + \frac{22981}{540} t^3 h - \frac{331681}{2880} t^4 h + \frac{16807}{90} t^5 h - \frac{386561}{2160} t^6 h + \frac{16807}{180} t^7 h - \frac{117649}{5760} t^8 h$$

$$\beta_{\frac{1}{7}} = \frac{49}{2} t^2 h - \frac{10927}{60} t^3 h + \frac{109417}{180} t^4 h - \frac{88837}{80} t^5 h + \frac{991613}{864} t^6 h - \frac{50421}{80} t^7 h + \frac{823543}{5760} t^8 h$$

$$\beta_{\frac{2}{7}} = -\frac{147}{4} t^2 h + \frac{14357}{40} t^3 h - \frac{1347647}{960} t^4 h + \frac{170471}{60} t^5 h - \frac{50421}{16} t^6 h + \frac{218491}{120} t^7 h - \frac{823543}{1920} t^8 h$$

$$\beta_{\frac{3}{7}} = \frac{245}{6} t^2 h - \frac{46501}{108} t^3 h + \frac{133427}{72} t^4 h - \frac{2926819}{720} t^5 h + \frac{4151329}{864} t^6 h - \frac{420175}{144} t^7 h + \frac{823543}{1152} t^8 h$$

$$\beta_{\frac{4}{7}} = -\frac{245}{8} t^2 h + \frac{2009}{6} t^3 h - \frac{872935}{576} t^4 h + \frac{52822}{15} t^5 h - \frac{1899191}{432} t^6 h + \frac{16807}{6} t^7 h - \frac{823543}{1152} t^8 h$$

$$\beta_{\frac{5}{7}} = \frac{147}{10} t^2 h - \frac{3283}{20} t^3 h + \frac{22981}{30} t^4 h - \frac{88837}{48} t^5 h + \frac{386561}{160} t^6 h - \frac{386561}{240} t^7 h + \frac{823543}{1920} t^8 h$$

$$\beta_{\frac{6}{7}} = -\frac{49}{12} t^2 h + \frac{49931}{1080} t^3 h - \frac{634207}{2880} t^4 h + \frac{98441}{180} t^5 h - \frac{319333}{432} t^6 h + \frac{184877}{360} t^7 h - \frac{823543}{5760} t^8 h$$

$$\beta_1 = \frac{1}{2} t^2 h - \frac{343}{60} t^3 h + \frac{9947}{360} t^4 h - \frac{16807}{240} t^5 h + \frac{84035}{864} t^6 h - \frac{16807}{240} t^7 h + \frac{117649}{5760} t^8 h$$

where  $t = \frac{x - x_n}{h}$ .

Evaluating (2.3) at  $t = \frac{1}{7} \left( \frac{1}{7} \right)_1$  gives a discrete block method of the form,

$$y_{n+\frac{1}{7}} = y_n + \frac{751}{17280} h f_n + \frac{139849}{846720} h f_{n+\frac{1}{7}} - \frac{4511}{31360} h f_{n+\frac{2}{7}} + \frac{123133}{846720} h f_{n+\frac{3}{7}} - \frac{88547}{846720} h f_{n+\frac{4}{7}} \\ + \frac{1537}{31360} h f_{n+\frac{5}{7}} - \frac{11351}{846720} h f_{n+\frac{6}{7}} + \frac{275}{169344} h f_{n+1}$$

$$y_{n+\frac{2}{7}} = y_n + \frac{41}{980} h f_n + \frac{1466}{6615} h f_{n+\frac{1}{7}} - \frac{71}{2940} h f_{n+\frac{2}{7}} + \frac{68}{735} h f_{n+\frac{3}{7}} - \frac{1927}{26460} h f_{n+\frac{4}{7}} \\ + \frac{26}{735} h f_{n+\frac{5}{7}} - \frac{29}{2940} h f_{n+\frac{6}{7}} + \frac{8}{6615} h f_{n+1}$$

$$y_{n+\frac{3}{7}} = y_n + \frac{265}{6272} h f_n + \frac{1359}{6272} h f_{n+\frac{1}{7}} + \frac{1377}{31360} h f_{n+\frac{2}{7}} + \frac{5927}{31360} h f_{n+\frac{3}{7}} - \frac{3033}{31360} h f_{n+\frac{4}{7}} \\ + \frac{1377}{31360} h f_{n+\frac{5}{7}} - \frac{373}{31360} h f_{n+\frac{6}{7}} + \frac{9}{6272} h f_{n+1}$$

$$y_{n+\frac{4}{7}} = y_n + \frac{278}{6615} h f_n + \frac{1448}{6615} h f_{n+\frac{1}{7}} + \frac{8}{245} h f_{n+\frac{2}{7}} + \frac{1784}{6615} h f_{n+\frac{3}{7}} - \frac{106}{6615} h f_{n+\frac{4}{7}} \\ + \frac{8}{245} h f_{n+\frac{5}{7}} - \frac{64}{6615} h f_{n+\frac{6}{7}} + \frac{8}{6615} h f_{n+1}$$

$$y_{n+\frac{5}{7}} = y_n + \frac{265}{6272} h f_n + \frac{36725}{169344} h f_{n+\frac{1}{7}} + \frac{775}{18816} h f_{n+\frac{2}{7}} + \frac{4625}{18816} h f_{n+\frac{3}{7}} + \frac{13625}{169344} h f_{n+\frac{4}{7}} \\ + \frac{1895}{18816} h f_{n+\frac{5}{7}} - \frac{275}{18816} h f_{n+\frac{6}{7}} + \frac{275}{169344} h f_{n+1}$$

$$y_{n+\frac{6}{7}} = y_n + \frac{41}{980} h f_n + \frac{54}{245} h f_{n+\frac{1}{7}} + \frac{27}{980} h f_{n+\frac{2}{7}} + \frac{68}{245} h f_{n+\frac{3}{7}} + \frac{27}{980} h f_{n+\frac{4}{7}} \\ + \frac{54}{245} h f_{n+\frac{5}{7}} + \frac{41}{980} h f_{n+\frac{6}{7}}$$

$$y_{n+1} = y_n + \frac{751}{17280} h f_n + \frac{3577}{17280} h f_{n+\frac{1}{7}} + \frac{49}{640} h f_{n+\frac{2}{7}} + \frac{2989}{17280} h f_{n+\frac{3}{7}} + \frac{2989}{17280} h f_{n+\frac{4}{7}} \\ + \frac{49}{640} h f_{n+\frac{5}{7}} + \frac{3577}{17280} h f_{n+\frac{6}{7}} + \frac{751}{17280} h f_{n+1}$$

### III. ANALYSIS OF THE CHEBYSHEV METHODS

This section focused on analyzing the order, error constant, consistency, zero-stability and region of absolute stability of Chebyshev polynomial.

#### A. ORDER AND ERROR CONSTANT

According to Sunday et al. (2015) on the Chebyshev method, the order of the Chebyshev method is  $p = [8 \ 8 \ 8 \ 8 \ 8 \ 8 \ 8]$ . That is, the Chebyshev method is of uniform order 8. The error constant is given by

$$C_9 = [-2.3186 \times 10^{-10}, -1.8203 \times 10^{-10}, -2.0411 \times 10^{-10}, -1.8706 \times 10^{-10}, -2.0914 \times 10^{-10}, -1.5931 \times 10^{-10}, -3.9117 \times 10^{-10}]^T.$$

#### B. CONSISTENCY

According to Sunday et al. (2015), the Chebyshev method is consistent due to its order  $p \geq 1$ . Consistency plays a crucial role in regulating the magnitude of the local truncation error at every computational stage.

#### C. ZERO STABILITY

According to Sunday et al. (2015), the first characteristic polynomial is expressed as follows:

$$\rho(z) = z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & z & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & z & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & z & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & z & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & z & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-z \end{bmatrix}$$

$$= z^6(z-1) = 0$$

Solving for  $z$  in

$$z^6(z-1) = 0 \quad (3.12)$$

Gives  $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0$  and  $z_7 = 1$ . Hence, the Chebyshev method in equation (3.9) is zero-stable.

#### D. CONVERGENCE

Based on Dalquist theorem in Yunusa et al. (2024), the Chebyshev method is convergent because it satisfies both consistency and zero-stability criteria.

#### E. REGION OF ABSOLUTE STABILITY

Employing the boundary locus method according to Sunday et al. (2015), we derive the stability polynomial for the Chebyshev method as

$$\bar{h}(w) = \left( -\frac{4507}{2529924096} w^7 - \frac{69371950261}{187437016424448000} w^6 \right) h^7 + \left( -\frac{331024314991}{44627861053440000} w^6 + \frac{273871}{75897722880} w^7 \right) h^6 + \left( -\frac{163985805373}{22222} w^6 - \frac{213337}{553190400} w^7 \right) h^5 + \left( -\frac{4118046305983}{2007417323520000} w^6 + \frac{10030753}{6638284800} w^7 \right) h^4 + \left( \frac{579227854099}{133827821568000} w^6 - \frac{3462563}{135475200} w^7 \right) h^3 + \left( -\frac{2099940793}{14936140800} w^6 + \frac{14137079}{135475200} w^7 \right) h^2 + \left( \frac{93379}{282240} w^6 - \frac{1}{2} w^6 \right) h$$

$$(3.13)$$

The depiction of the region of absolute stability for the Chebyshev method is illustrated as

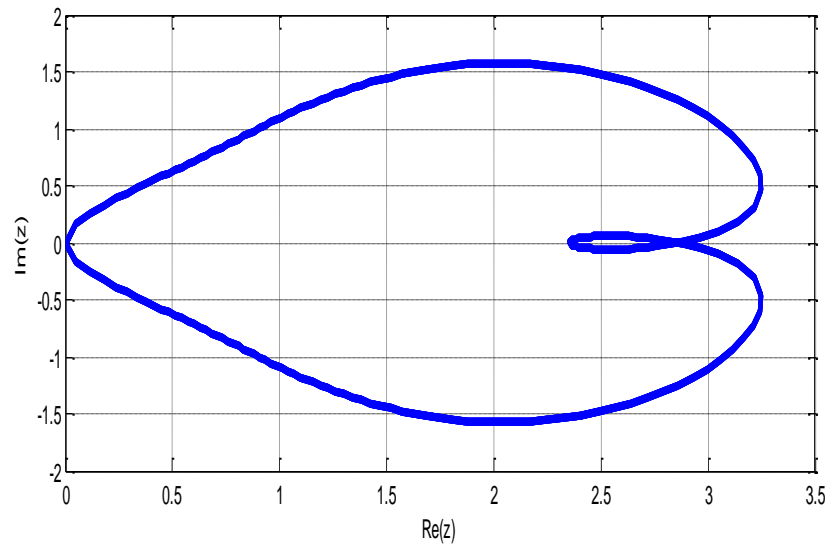


Figure 1: Stability region of Chebyshev method.

#### IV. NUMERICAL APPLICATION

The newly derived block methods shall be applied on some oscillatory problems of the form (1.1) as presented below. On the application of new methods on some oscillatory differential equations (among others Malthus Growth Model, Prothrethorath differential equation and highly stiff oscillatory problems). The absolute error obtained from new methods were compare with the existing methods.

##### Application Problem 4.1 (Malthus Growth Model)

The Malthusian growth model, named after British economist Thomas Robert Malthus, is a theory of population growth that highlights the potential for exponential population increase and its implications for resources. Malthus first presented his ideas in his 1798 work, "An Essay on the Principle of Population." At its core, the model posits that populations tend to grow exponentially when resources are abundant, meaning that the population size can double at a constant rate over successive time periods, assuming no constraints. This exponential growth of the population starkly contrasts with the arithmetic growth of food supply or resources, which Malthus believed increased by a fixed amount each period rather than multiplying. The Malthusian theory explained that the human population grows more rapidly than the food supply until famines, wars, or diseases reduce the population (Oluwaseun and Zurni (2022).

The Malthusian growth model can be described using a simple differential equation:

$$\frac{dy}{dx} = kp, \quad x \in [0,1], \quad (4.1)$$

with the exact solution given by

$$y(x) = 100 \exp(0.250679566129x) \quad (4.2)$$



Initial condition  $y(0)=100$  with  $k = 0.250679566129$  and  $h = 0.1$

Source: (Oluwaseun and Zurni (2022); Yunusa et al. (2024).

**Application Problem 4.2: (Prothero-Robinson Equation)**

Take into account the Prothero-Robinson oscillatory differential equation, which has been addressed by Sunday et al. (2014) and Sunday et al. (2015), formulated as follows:

$$y' = \Phi(y - \sin x) - y, \Phi = -1, y(0) = 0 \quad (4.3)$$

Which has the exact solution as

$$y(x) = \sin x \quad (4.4)$$

**Application Problem 4.3: (Oscillatory Differential Equation)**

James et al. (2013) and Sunday et al. (2015) investigate the highly stiff oscillatory differential equation presented below:

$$y' = x - y, y(0) = 0, 0 \leq x \leq 1 \quad (4.5)$$

Whose exact solution is given as

$$y(x) = x + \exp(-x) - 1 \quad (4.6)$$

Source: (James et al. (2013); Sunday et al. (2015)).

**Application Problem 4.4: (Oscillatory Differential Equation)**

Examining the oscillatory differential equation

$$y' = xy, y(0) = 1, 0 \leq x \leq 1, h = 0.1 \quad (4.7)$$

Which possesses a solution that can be expressed analytically

$$y(x) = \exp\left(\frac{1}{2}x^2\right) \quad (4.8)$$

Source: (James et al., 2013, Sunday et al., 2015).

**Table 1:** The results of application problem 1 with Oluwaseun and Zurni (2022), Yunusa et al., (2024).

x	Exact Solution	Computed Solution	Absolute Errors	Oluwaseun and Zurni (2022)	Yunusa et al. (2024)
0.1	102.53847998347329794000	102.53847998347329792000	2.0000(-17)	1.6677(-08)	0.0000(00)
0.2	105.14139877321154182000	105.14139877321154181000	1.0000(-17)	4.4003(-10)	0.0000(00)
0.3	107.81039213541335645000	107.81039213541335643000	2.0000(-17)	1.7117(-08)	0.0000(00)
0.4	110.54713735987489512000	110.54713735987489510000	2.0000(-17)	8.8005(-10)	0.0000(00)
0.5	113.35335431405805132000	113.35335431405805130000	2.0000(-17)	1.7557(-08)	1.4211(-14)
0.6	116.23080652391598100000	116.23080652391598096000	4.0000(-17)	1.3201(-09)	1.4211(-14)
0.7	119.18130228215516429000	119.18130228215516425000	4.0000(-17)	1.7997(-08)	1.4211(-14)
0.8	122.20669578463047796000	122.20669578463047791000	5.0000(-17)	1.7601(-09)	0.0000(00)
0.9	125.30888829558742918000	125.30888829558742914000	4.0000(-17)	1.8437(-08)	1.4211(-14)
1.0	128.48982934248383035000	128.48982934248383028000	7.0000(-17)	2.2001(-09)	0.0000(00)

**Table 2:** The results of application problem 2 with Sunday et al. (2014), Sunday et al. (2015)

x	Exact Solution	Computed Solution	Absolute Errors	Sunday et al. (2014)	Sunday et al. (2015)
0.1	0.09983341664682815231	0.09983341664682815265	3.4000(-19)	1.4530(-11)	1.3422(-11)
0.2	0.19866933079506121546	0.19866933079506121613	6.7000(-19)	1.6211(-11)	2.1464(-11)
0.3	0.29552020666133957511	0.29552020666133957607	9.6000(-19)	2.1310(-11)	3.2359(-11)
0.4	0.38941834230865049167	0.38941834230865049287	1.2000(-18)	1.3799(-11)	4.1877(-11)
0.5	0.47942553860420300027	0.47942553860420300171	1.4400(-18)	2.7441(-11)	4.6377(-11)
0.6	0.56464247339503535720	0.56464247339503535885	1.6500(-18)	1.1114(-11)	5.3368(-11)
0.7	0.64421768723769105367	0.64421768723769105550	1.8300(-18)	2.8657(-11)	5.8936(-11)
0.8	0.71735609089952276163	0.71735609089952276357	1.9400(-18)	1.9218(-10)	6.0221(-11)
0.9	0.78332690962748338846	0.78332690962748339049	2.0300(-18)	1.2392(-10)	6.3342(-11)
1.0	0.84147098480789650665	0.84147098480789650874	2.0900(-18)	1.4711(-10)	6.5059(-11)

**Table 3:** The results of application problem 3 with James et al. (2013) and Sunday et al. (2015)

X	Exact Solution	Computed Solution	Absolute Errors	Error in Sunday et al. (2015)	Error James et al. (2013)
0.1	0.00483741803595957320	0.00483741803595957281	3.9000(-19)	1.0899(-14)	1.6554(-11)
0.2	0.01873075307798185870	0.01873075307798185803	6.7000(-19)	3.6577(-14)	4.3981(-11)
0.3	0.04081822068171786610	0.04081822068171786519	9.1000(-19)	4.4761(-14)	7.8451(-11)
0.4	0.07032004603563930070	0.07032004603563929969	1.0100(-18)	6.1209(-14)	1.2662(-11)
0.5	0.10653065971263342360	0.10653065971263342241	1.1900(-18)	6.1209(-14)	1.9709(-10)
0.6	0.14881163609402643260	0.14881163609402643133	1.2700(-18)	7.0592(-14)	3.0180(-10)
0.7	0.19658530379140951470	0.19658530379140951333	1.3700(-18)	7.9268(-14)	4.5771(-10)
0.8	0.24932896411722159140	0.24932896411722159001	1.3900(-18)	8.3601(-15)	6.8954(-09)
0.9	0.30656965974059911190	0.30656965974059911044	1.4600(-18)	9.4146(-15)	1.0336(-09)
1.0	0.36787944117144232160	0.36787944117144232016	1.4400(-18)	9.7071(-15)	1.5435(-09)

**Table 4:** The results of application problem 4 with James et al. (2013), Sunday et al. (2015)

X	Exact Solution	Computed Solution	Absolute Errors	James, et al. (2013)	Sunday et al. (2015)
0.1	1.00501252085940106340	1.00501252085940108180	1.8400(-17)	1.6554(-11)	1.2473(-13)
0.2	1.02020134002675581020	1.02020134002675588710	7.6900(-17)	4.3981(-11)	2.4989(-13)
0.3	1.04602785990871694270	1.04602785990871712630	1.8360(-16)	7.8451(-11)	4.0149(-13)
0.4	1.08328706767495855440	1.08328706767495890830	3.5390(-16)	1.2662(-11)	5.7196(-13)
0.5	1.13314845306682631680	1.13314845306682692950	6.1270(-16)	1.9709(-10)	7.5116(-13)
0.6	1.19721736312181016490	1.19721736312181116270	9.9780(-16)	3.0180(-10)	9.2698(-13)
0.7	1.27762131320488661070	1.27762131320488817810	1.5674(-15)	4.5771(-10)	3.0572(-12)
0.8	1.37712776433595708450	1.37712776433595949320	2.4087(-15)	6.8954(-09)	3.1135(-12)
0.9	1.49930250005676686970	1.49930250005677052230	3.6526(-15)	1.0336(-09)	6.1995(-12)
1.0	1.64872127070012814680	1.64872127070013364430	5.4975(-15)	1.5435(-09)	6.6348(-12)

## V. DISCUSSION OF RESULTS

The Malthusian growth model represents exponential population growth and serves as an important example of how mathematical modeling can explain real-world phenomena. Table 4.1 evaluates the computed solutions against the exact solutions using different numerical methods. The absolute errors, though extremely small, highlight the level of accuracy attained by the methods under consideration. In particular, the methods proposed by Oluwaseun and Zurni (2022) and Yunusa et al. (2024) show impressive precision.

Additionally, the error patterns observed in the table suggest a consistent behavior across different intervals, indicating stability in the computational approach. The comparison between Oluwaseun and Zurni (2022) and Yunusa et al. (2024) emphasizes the iterative improvement in numerical techniques over time. This consistency underlines the robustness of these methods when applied to real-world problems. Table 4.1 confirms the effectiveness of contemporary numerical methods in solving exponential growth models. It underscores the relevance of continuous advancements in numerical solutions to enhance accuracy and reliability, particularly in applications involving sensitive systems such as population dynamics.

The Prothero-Robinson equation presents a more oscillatory nature compared to the Malthusian growth model. Table 4.2 compares the exact solutions and computed solutions, highlighting the precision of methods by Sunday et al. (2014) and Sunday et al. (2015). Both methods achieve extremely low absolute errors, affirming their reliability in solving oscillatory equations. Notably, the error values reveal slight variations between the two methods, with the later method demonstrating improved accuracy in some cases. Table 4.2 underscores the effectiveness of modern numerical methods in handling oscillatory systems. It demonstrates their practical utility in modeling phenomena like wave propagation or electrical oscillations, where precision is paramount.

Stiff oscillatory equations, such as the one addressed in Table 4.3, pose unique challenges for numerical solvers due to their sensitivity to small perturbations and rapid changes in behavior. The table illustrates the performance of numerical methods developed by James et al. (2013) and Sunday et al. (2015), both of which achieve remarkable accuracy. However, slight differences in error magnitudes indicate that Sunday et al.'s method is better suited for managing the stiffness of the problem. Table 4.3 showcases the importance of specialized numerical methods for addressing stiff oscillatory equations. It highlights the value of precision, stability, and iterative refinement in tackling challenging problems effectively.

The oscillatory equation analyzed in Table 4.4 introduces additional complexity due to its extended oscillatory behavior. The results compare numerical approximations using methods by James et al. (2013) and Sunday et al. (2015), both of which achieve high precision with negligible absolute errors. However, the table reveals that Sunday et al. (2015) achieves slightly lower error magnitudes. Table 4.4 highlights the practical utility of refined numerical methods for solving oscillatory equations. It underscores the importance of continuous research in numerical analysis to ensure the accuracy and stability of solutions for complex dynamic systems.

## V. SUMMARY AND CONCLUSION

The study focused on the development and analysis of numerical methods for solving first-order differential equations using Chebyshev polynomials. Emphasis was placed on deriving methods based on the Chebyshev polynomial of the first kind, leveraging their orthogonality and approximation properties. The continuous linear multistep methods (LMMs) were developed through interpolation and collocation techniques, resulting in schemes with high accuracy and computational efficiency. The order, error constants, consistency, zero stability, and convergence of the methods were rigorously analyzed. The new methods were applied to solve oscillatory problems, including the Malthusian growth model and highly stiff oscillatory equations, with comparisons made against existing methods. Results demonstrated the superiority of the Chebyshev-based approach in terms of precision and reduced computational error.

The study concludes that numerical methods based on Chebyshev polynomials of the first kind provide robust tools for solving first-order ordinary differential equations. The derived methods exhibit uniform order, high accuracy, and excellent stability properties, making them suitable for practical applications, especially in solving oscillatory and stiff differential equations. Comparisons with existing methods revealed that the Chebyshev-based techniques achieved better accuracy and efficiency. The findings underscore the potential of leveraging orthogonal polynomials in advancing numerical analysis for differential equations. Future work may explore further extensions to higher-order problems and adaptive step-size methods for broader applicability.

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