

# L-stable Extended One-Step Hybrid Block Linear Multistep Methods for Solving Nonlinear Semi-Explicit Hessenberg DAEs Indexes with a Wider Range

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**ABSTRACT** Differential algebraic equations (DAEs) are natural extension of differential equations that also involve algebraic equations. Several numerical methods have been developed to solve these DAEs, mostly for low-index systems. The methods are RKMs and others. SEHDAEs are important type of DAEs that are commonly used in the modeling of mechanical systems and electrical circuits. These methods either lack a rigorous theoretical analysis for higher-index systems. The LMM preserves the self-starting characteristics over existing methods and also offers the advantage of approximating the solution at multiple points in which the existing numerical and analytical methods cannot perform. EHBBDF of order nine was presented in one step for the numerical solution of SEHDAEs of lower and higher index. The main objectives of this work are to develop a new EHBBDF, investigate numerical properties, and compare the numerical results with existing methods for the general solution of SEHDAEs and it was carried out with MAPLE 2016 software. Collocation and interpolation techniques are used to construct the approach which yields the main and supplementary techniques. The formula is then used to create a single block of methods that simultaneously provide the approximate solutions for the SEHDAEs of lower and higher indexes. The new method (EHBBDF) was revealed consistent, convergent, and L-stable. The effectiveness and precision of the novel approach are assessed using four numerical examples. In specific, all differential variables of nonlinear SEHDAEs suggests the potential for extension of index 4 or higher.

**KEYWORDS:** Backward Differentiation Formulae, Semi- Explicit Hessenberg Differential-Algebraic Equations, Extended Hybrid Block, Rate of Convergence, Stability Analysis

## I. INTRODUCTION

Explicit Hessenberg semi-direct Differential algebraic equations (EHSDAEs) are used in many scientific and technical fields. They are a blend of differential and algebraic equations. These include of circuit analysis, CAD, electrical networks, dynamical systems, chemical reactions, mechanical or multibody systems, and optimum control. Several authors have developed numerical methods to render the solutions of the DAEs of the form

$$\left. \begin{aligned} F(t, x(t), x'(t), \dots, x^n(t)) &= 0 \\ G(t, x_1(t), x_2(t) \dots x_i(t)) &= 0 \end{aligned} \right\} \quad (1)$$

It is possible to convert certain systems to zero-index DAE or ODE systems, which may lessen the sparseness of Jacobian matrices. Then, these systems are solved directly (see Michael, Roswitha, Caren Ewa, and Stefan (2017); Paolo and Pierangelo (2011)). One of the most important ideas in numerical

solutions of DAEs is the index of a DAE. This shows how far away the DAE is from becoming an ODE, in a sense. It is well known that differential algebraic equations (DAEs) may be more difficult to solve if their index is bigger, that is, more than 1. Campbell, Petzold, and Brenan (1989). The backward differentiation formula (BDF) is the first generic technique used for the numerical solution of DAEs.

Since then, a number of changes and novel approaches have been proposed and investigated; see Ascher, Chin, and Reich (1994); Ascher, Chin, Petzold, and Reich (1995), as well as the cited works, for a discussion of these. An alternative approach is called "regularisation of DAEs," which entails supplementing the original DAE with a modest perturbation term (which is indicated by a small positive parameter) (see Hanke, 1992; Petzold; Ren and Maly (1993); Kalachev and O'Malley (1995); and Knorrenschild (1992). utilising semi-analytical techniques like the Padé's approximations (see Ascher & Petzold, 1999), adomian decomposition methods (ADMs) (see Biazar, Babolian, & Islam, 2003; Biazar, Tango, Babolian, & Islam, 2004; Donalchili & Jafari, 2020), variational iterative methods (VIMs), pseudospectral methods (see Vazquez-Leall and Sarmiento-Reyes (2015)), differential transform methods (DTM) (see Brahim Benhammouda, 2023), variational formulation techniques (VFT) (see Radhi Ali and Ghazwa Faisal, (2015); Radhi A. Zaboon, Ghazwa F. Abd, (2021)). For the numerical solution of IVPs, Ebadi and Gokhale (2010) and Ebadi and Gokhale (2011) introduced class2+1 hybrid BDF-like techniques, hybrid BDF methods (HBDF), and novel hybrid methods. In comparison to the extended BDF (EBDF) and modified extended BDF (MEBDF) approaches, these techniques have broad stability areas and perform well when it comes to CPU time (Cash, 2000). The benefit of approximating the solution of (1) at several sites is provided by the block approach, which maintains the self-starting features of the Runge-Kutta technique (see Akinfenwa & Okunuga, 2014). Inspired by Akinnukawe et al.'s (2019) study, which suggested block hybrid approaches for DAEs. To solve the differential-algebraic equations (1), this paper proposes an Extended Hybrid Block Second Derivative Backward Differentiation Formulae that are not only self-starting but also have good accuracy, stability properties and are less expensive in terms of the number of functions evaluated per step.

## II. METHODOLOGY

### A. CONSTRUCTION OF EHBDF

The newly hybrid block method EHBDF on the interval from  $t_n$  to  $t_{n+1} = t_n + h$ , is developed and used to obtain implicit continuous scheme of the main and complementary techniques of extended hybrid block backward differentiation formulae (EHBDF), where  $h$  is the selected step length and  $k$  is the method's step size. It is assumed that  $Y(t)$  is a local representation of the exact solution  $y(t)$  on the interval  $[t_n, t_{n+1}]$ , which is given by

$$Y(t) = \sum_{j=0}^{m+n-1} w_j \vartheta_j(t) \tag{3}$$

$w_j$  are unknown coefficients to be determined, and  $\vartheta_j(t)$  are polynomial basis function of degree  $m + n - 1$  such that the number of interpolation points  $m$  and the number of distinct collocation points  $n$  are respectively chosen to satisfy  $1 \leq m \leq k, k = 2$  and  $n > 0$ . The proposed class of methods is thus constructed by imposing the following conditions

$$\sum_{j=0}^{4k+5} w_j t_{n+i}^j = y_{n+i}, \quad i = 0, \frac{1}{2}, 1 \tag{4}$$

$$\sum_{j=0}^{4k+5} w_j t_{n+i}^j = f_{n+i}, \quad i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \tag{5}$$

$$\sum_{j=0}^{4k+5} w_j t_{n+i}^j = g_{n+i}, \quad i = \frac{1}{2}, 2 \tag{6}$$

Assuming that  $y_{n+i} = Y(t_n + ih)$ , denote the numerical approximation to the exact solution  $y(t_{n+i})$ :  $f_{n+i} = Y'(t_n + ih)$ , and  $g_{n+i} = Y''(t_n + ih)$  indicate the approximation to  $y''(t_{n+i})$ ,  $n$  is the grid index. It should be noted that equations (4), (5), and (6) leads to a system of  $4k + 5$  equations that must be solved to obtain the coefficients  $w_j$ . The extended hybrid block backward differentiation formula (EHBBDF) is then obtained by substituting the values of  $w_j$  into equation (3) after some algebraic reduction, the EHBBDF yields

$$Y(x) = \sum_{j=0}^2 \left( \alpha_j(x) y_{n+\frac{j}{2}} \right) + h \left( \sum_{i=0}^4 \beta_i(x) f_{n+\frac{i}{2}} \right) + h^2 \left( \gamma_{\frac{\mu}{2}}(x) g_{n+\frac{\mu}{2}} + \gamma_{k+1}(x) g_{n+k+1} \right) \quad (7)$$

Where

$\alpha_j(x), j = 0(1)2; \beta_i(x), i = 0(1)4; \gamma_{\frac{\mu}{2}}(x), \gamma_{k+1}(x)$  are continuous coefficients.  $\mu = 1, k = 1$  is step number. The continuous form of EHBBDF (7) is then used to generate the main discrete method by evaluating at point  $x = x_{n+k+1}$  and  $x = x_{n+k+\frac{1}{2}}$  to yield (8) in order to obtain extended hybrid block method.

$$\left. \begin{aligned} y_{n+2} &= \frac{h^2}{5729} \left( -78g_{n+2} - 768g_{n+\frac{1}{2}} \right) + \frac{h}{5729} \left\{ 270f_n - 1296f_{n+1} + 1236f_{n+2} \right. \\ &\quad \left. - 3456f_{n+\frac{1}{2}} + 3456f_{n+\frac{3}{2}} \right\} + \frac{2673}{5729} y_n - \frac{16384}{5729} y_{n+\frac{1}{2}} + \frac{19440}{5729} y_{n+1} \\ y_{n+\frac{3}{2}} &= \frac{h^2}{45832} \left( 54g_{n+2} + 5820g_{n+\frac{1}{2}} \right) + \frac{h}{45832} \left\{ -1509f_n + 48492f_{n+1} - 415f_{n+2} \right. \\ &\quad \left. + 37648f_{n+\frac{1}{2}} + 8184f_{n+\frac{3}{2}} \right\} + \frac{1939}{5729} y_n - \frac{21249}{5729} y_{n+\frac{1}{2}} - \frac{13581}{5729} y_{n+1} \end{aligned} \right\} \quad (8)$$

To Obtain the additional method, we differentiate continuous form (CF) (7) twice with respect to  $x$ , then we obtain (9)

$$y''(x) = \frac{1}{h^2} \left\{ \sum_{j=0}^2 \left( \alpha_j''(x) y_{n+\frac{j}{2}} \right) + h \left( \sum_{i=0}^4 \beta_i''(x) f_{n+\frac{i}{2}} \right) + h^2 \left( \gamma_{\frac{\mu}{2}}''(x) g_{n+\frac{\mu}{2}} + \gamma_{k+1}''(x) g_{n+k+1} \right) \right\} \quad (9)$$

By collocating (9) at some specific point steps  $x_{n+\frac{i}{2}}$ , where  $i = 2, 3$  twice, the complementary method is given as (10) below

$$\left. \begin{aligned} h^2 g_{n+1} &= -\frac{h^2}{309366} \left( 1272g_{n+2} + 350976g_{n+\frac{1}{2}} \right) + \frac{h}{309366} \left\{ -71829f_n + 3067200f_{n+1} \right. \\ &\quad \left. - 9139f_{n+2} + 2862688f_{n+\frac{1}{2}} + 70560f_{n+\frac{3}{2}} \right\} - \frac{233268}{5729} y_{n+1} + \frac{247296}{5729} y_{n+\frac{1}{2}} - \frac{14028}{5729} y_n \\ h^2 g_{n+\frac{3}{2}} &= -\frac{h^2}{17187} \left\{ -71427g_{n+2} - 1212g_{n+\frac{1}{2}} \right\} + \frac{h}{17187} \left\{ 20501f_n - 400896f_{n+1} + \right. \\ &\quad \left. 9951f_{n+2} - 415950f_{n+\frac{1}{2}} + 49294f_{n+\frac{3}{2}} \right\} + \frac{560772}{5729} y_{n+1} - \frac{630144}{5729} y_{n+\frac{1}{2}} + \frac{69372}{5729} y_n \end{aligned} \right\} \quad (10)$$

Hence, combining equation (8) and (10) to form extended hybrid block method for  $k = 1$  below

$$\left. \begin{aligned}
 y_{n+2} &= \frac{h^2}{5729} \left( -78g_{n+2} - 768g_{n+\frac{1}{2}} \right) + \frac{h}{5729} \left\{ 270f_n - 1296f_{n+1} + 1236f_{n+2} \right. \\
 &\quad \left. - 3456f_{n+\frac{1}{2}} + 3456f_{n+\frac{3}{2}} \right\} + \frac{2673}{5729}y_n - \frac{16384}{5729}y_{n+\frac{1}{2}} + \frac{19440}{5729}y_{n+1} \\
 y_{n+\frac{3}{2}} &= \frac{h^2}{45832} \left( 54g_{n+2} + 5820g_{n+\frac{1}{2}} \right) + \frac{h}{45832} \left\{ -1509f_n + 48492f_{n+1} - 415f_{n+2} \right. \\
 &\quad \left. + 37648f_{n+\frac{1}{2}} + 8184f_{n+\frac{3}{2}} \right\} + \frac{1939}{5729}y_n - \frac{21249}{5729}y_{n+\frac{1}{2}} - \frac{13581}{5729}y_{n+1} \\
 h^2g_{n+1} &= -\frac{h^2}{309366} \left( 1272g_{n+2} + 350976g_{n+\frac{1}{2}} \right) + \frac{h}{309366} \left\{ -71829f_n + 3067200f_{n+1} \right. \\
 &\quad \left. - 9139f_{n+2} + 2862688f_{n+\frac{1}{2}} + 70560f_{n+\frac{3}{2}} \right\} - \frac{233268}{5729}y_{n+1} + \frac{247296}{5729}y_{n+\frac{1}{2}} - \frac{14028}{5729}y_n \\
 h^2g_{n+\frac{3}{2}} &= -\frac{h^2}{17187} \left\{ -71427g_{n+2} - 1212g_{n+\frac{1}{2}} \right\} + \frac{h}{17187} \left\{ 20501f_n - 400896f_{n+1} + \right. \\
 &\quad \left. 9951f_{n+2} - 415950f_{n+\frac{1}{2}} + 49294f_{n+\frac{3}{2}} \right\} + \frac{560772}{5729}y_{n+1} - \frac{630144}{5729}y_{n+\frac{1}{2}} + \frac{69372}{5729}y_n
 \end{aligned} \right\} \quad (11)$$

**B. ANALYSIS OF THE EHBBDF**

The primary numerical characteristics and degree of accuracy of the approach, including Local Truncation Error (LTE), Consistency, Zero Stability, Convergence, Linear Stability, and the area of absolute stability of the methods, were verified by analysing the aforementioned EHBBDF. The EHBBDF can be represented by a matrix finite difference equation in block form as

$$U^{(1)}Y_{\omega+1} = U^{(0)}Y_{\omega} + h(B^{(1)}f_{\omega+1} + B^{(0)}f_{\omega}) + h^2(C^{(1)}g_{\omega+1}) \quad (12)$$

where

$$\left. \begin{aligned}
 Y_{w+1} &= \left( y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2} \right)^T, Y_w = \left( y_{n-\frac{1}{2}}, y_{n-1}, y_{n-\frac{3}{2}}, y_n \right)^T \\
 F_{w+1} &= \left( f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2} \right)^T, F_w = \left( f_{n-\frac{1}{2}}, f_{n-1}, f_{n-\frac{3}{2}}, f_n \right)^T \\
 G_{w+1} &= \left( g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{1}{2}}, g_{n+2} \right)^T, G_w = \left( g_{n-\frac{1}{2}}, g_{n-1}, g_{n-\frac{3}{2}}, g_n \right)^T
 \end{aligned} \right\}$$

$$\omega = 0, \frac{1}{2}, 1, \dots \text{ and } n = 0, 2, \dots, N - 2.$$

and the matrices  $U^{(1)}, U^{(0)}, B^{(1)}, B^{(0)}, C^{(1)}$  are 4 by 4 matrices whose entries in the coefficients of equation (12) are respectively defined as follows:

$$\begin{bmatrix} -\frac{247296}{5729} & \frac{233268}{5729} & 0 & 0 \\ \frac{630144}{5729} & -\frac{560772}{5729} & 0 & 0 \\ -\frac{21249}{5729} & \frac{13581}{5729} & 1 & 0 \\ \frac{16384}{5729} & -\frac{19440}{5729} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{14028}{5729} \\ 0 & 0 & 0 & -\frac{69372}{5729} \\ 0 & 0 & 0 & \frac{1939}{5729} \\ 0 & 0 & 0 & -\frac{2673}{5729} \end{bmatrix} \begin{bmatrix} -\frac{58496}{51561} & 1 & 0 & -\frac{212}{51561} \\ \frac{23809}{5729} & 0 & 1 & \frac{404}{5729} \\ -\frac{1455}{11458} & 0 & 0 & -\frac{27}{22916} \\ \frac{768}{5729} & 0 & 0 & \frac{78}{5729} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1431344}{154683} & \frac{56800}{5729} & \frac{3920}{17187} & \frac{9139}{309366} \\ \frac{138650}{5729} & \frac{133632}{5729} & -\frac{49294}{17187} & -\frac{3317}{5729} \\ -\frac{4706}{5729} & -\frac{12123}{11458} & -\frac{1023}{5729} & \frac{415}{45832} \\ \frac{3456}{5729} & \frac{1296}{5729} & -\frac{3456}{5729} & -\frac{1236}{5729} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{7981}{34374} \\ 0 & 0 & 0 & -\frac{20501}{17187} \\ 0 & 0 & 0 & \frac{1509}{45832} \\ 0 & 0 & 0 & -\frac{270}{5729} \end{bmatrix}$$

Where  $C^{(0)}$  is the zero matrix.

**C. ORDER AND ERROR CONSTANTS OF EHBBDF**

According to Fatunla (1991a) and Lambert (1991b), the local truncation error for each process in the EHBBDF may be defined by a linear difference operator.

$$\mathcal{L}\{y(x_n); h\} = \sum_{j=0}^2 \left( \alpha_{\frac{j}{2}} \cdot y_{n+\frac{j}{2}} \right) - \left[ h \sum_{j=0}^4 \left( \beta_{\frac{j}{2}} \cdot f_{n+\frac{j}{2}} \right) + h^2 \left\{ \gamma_{\frac{\mu}{2}} g_{n+\frac{\mu}{2}} + \gamma_{k+1} g_{n+k+1} \right\} \right] \quad (13)$$

The expression in equation (13) can be obtained by writing the terms as a Taylor series about the point t, provided that  $y(t)$  is sufficiently differentiable.

$$L\{y(x); h\} = c_0 y(x_n) + c_1 h y'(x_n) + h^2 c_2 y''(x_n) + \dots + c_p h^p y^{(p)}(x_n) + \dots \quad (14)$$

Where the constant  $c_p, p = 0,1,2, \dots$  are given as follows

$$c_0 = \sum_{j=0}^2 \alpha_{\frac{j}{2}}$$

$$c_1 = \sum_{j=1}^2 \left( \frac{j}{2} \alpha_{\frac{j}{2}} \right) - \sum_{j=1}^4 \left( \beta_{\frac{j}{2}} \right)$$

$$c_2 = \sum_{j=1}^2 \left( \frac{j^2}{2} \alpha_{\frac{j}{2}} \right) - \sum_{j=1}^4 \left( \frac{j}{2} \beta_{\frac{j}{2}} \right) - \gamma_{\frac{1}{2}} - \gamma_2$$

$$\vdots$$

$$c_p = \frac{1}{p!} \left[ \sum_{j=1}^{k+1} \left( \frac{j}{2} \right)^p \alpha_{\frac{j}{2}} \right] - \frac{1}{(p-1)!} \left[ \sum_{j=0}^{k+3} \left( \frac{j}{2} \right)^{p-1} \beta_{\frac{j}{2}} \right] + \frac{1}{(p-2)!} \left\{ \gamma_{\frac{\mu}{2}} \cdot \left( \frac{\mu}{2} \right)^{p-2} + \gamma_{k+1} \cdot (k+1)^{p-2} \right\}$$

According to Henrici (1962), we say that the methods in (11) have a maximal order of accuracy  $p$  if:

$$L\{y(x); h\} = c_{p+1}h^{p+1}y^{p+1}(x_n) + O(h^{p+2}), c_0 = c_1 = c_2 \dots c_p = 0, c_{p+1} \neq 0 \quad (15)$$

Therefore  $c_{p+1}$  is the error constant and  $c_{p+1}, h^{p+1}, y^{p+1}(x_n)$  the principal local truncation error of each algorithm in each hybrid block mentioned is calculated using (11). Then, the value of the error constant and Order of 1-step extended hybrid block BDF-type methods are presented below.

$$\left\{ \frac{1}{36665600}, \frac{-577}{49278566400}, \frac{18001}{369589244800}, \frac{-9743}{166315161600} \right\}^T \text{ and } \{9, 9, 9, 9\}^T$$

Where T is transpose

#### D. ZERO STABILITY

The zero stability of the method is concerned with the stability of the difference system in the limit as  $h$  tends to zero see Fatunla (1991). Thus, as  $h \rightarrow 0$  the difference system (12) tends to

$$U^{(1)}Y_{\omega+1} = U^{(0)}Y_{\omega}$$

According to Lambert (1973), any method is zero stable if and only if, the determinant of characteristics equations of the first matrices is equal to zero

$$\begin{aligned} \rho(R) = \det[RU^{(1)} - U^{(0)}] &= \frac{1451520}{5729} R^3(R - 1) = 0 \\ \Rightarrow |R| &= 0, 0, 0, 1 \end{aligned}$$

The EHBBDF is zero-stable for those roots with a multiplicity that does not exceed 1. Hence, the EHBBDF method is zero-stable when.

#### E. CONSISTENCY AND CONVERGENCE

The EHBBDF may be seen to be constant since its order  $p > 1$ . Thus, the EHBBDF (12) is zero-stable. Convergence, according to Henrici (1962), is consistency plus zero stability. This leads to the convergence of the EHBBDF.

#### F. LINEAR STABILITY

Following the approach of Hairer and Wanner (1996), the linear stability characteristics of the EHBBDF are analysed and determined by presenting them in the format (13) and employing the test problem.

$$y' = \lambda y, \quad y'' = \lambda^2 y, \quad \lambda < 0$$

To produce

$$Y_{\omega} = P(z)Y_{\omega+1}, \quad z = \lambda h \quad (16)$$

where the matrix  $P(z)$  is given by

$$P(z) = (A^{(1)} - zB^{(1)} - z^2C^{(1)})^{-1} \cdot (A^{(0)} + B^{(0)} + C^{(0)})$$

The matrix  $P(z)$  has eigenvalues  $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} = \{0, 0, 0, \varphi_4\}$ , where the dominant eigenvalue  $\varphi_4$ , is the stability function  $\varphi(z): \mathbb{C} \rightarrow \mathbb{C}$ , which is a rational function with real coefficients given by

$$P(z) = \frac{10854 z^7 + 138483 z^6 + 182109 z^5 - 1654087 z^4 - 2508567 z^3 + 7440496 z^2 + 7343472 z - 8709120}{19218z^8 + 76872z^7 - 194589z^6 - 852819z^5 - 5112969z^4 - 8714889z^3 - 2157936z^2 + 2044560z + 8709120} \quad (17)$$

The stability domain of the method (or region of absolute stability)  $S$  is defined as

$$P = \{z \in \mathbb{C}: \mathbb{R}(z) \leq 1\} \quad (18)$$

The approach is deemed A-stable if and only if  $P$  encompasses the left-half complex plane. The plotting of Figure 1 illustrates rectangles representing the blue and white colours for the regions of (18). The stability region, depicted in blue on the plot, is defined by the stability function (18). Given that there is no pole of

the stability function (18) in the left half of the complex plane, as noted by Hairer and Wanner (1996), it is clear from Figure 1 that our method is A-stable.

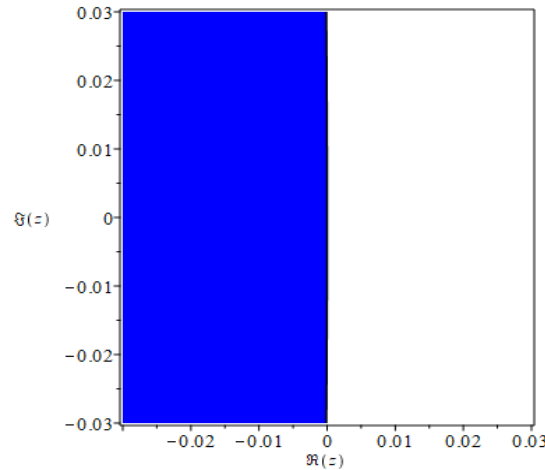


Figure 1: A Region of Absolute Stability (RAS) @  $k = 1$

### III. RESULTS AND DISCUSSION

#### A. COMPUTATIONAL ASPECT OF THE EHBDF

This section presents four examples to illustrate the method's accuracy. The four issues represent common challenges faced by DAEs, each carrying substantial implications for applied systems. We assess the maximum absolute error of the estimated solution. All computations were conducted using our proprietary Mathematica code within MAPLE 2016.

#### B. NUMERICAL RESULTS

The method is implemented more efficiently as a 1-step modified extended hybrid block BDF-type numerical integrator for (1) to simultaneously obtain the approximations  $\left\{y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\right\}^T$  without requiring back values or predictors to take  $n = 0, 4, \dots, N - 4$  over sub-intervals  $[x_0, x_4], \dots, [x_{N-4}, x_N]$ .

For example,  $n = 0, w = 1, \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}\right)^T$  are simultaneously obtained over the sub-interval  $[x_4, x_8]$ , as  $y_0$  is known from the initial value problem, are simultaneously obtained over the sub-interval, as is known from the previous block, and so on. The sub-intervals are non-overlapping as a consequence. The code utilises Newton's method for nonlinear problems and Gaussian elimination for linear problems.

##### a) Numerical Example

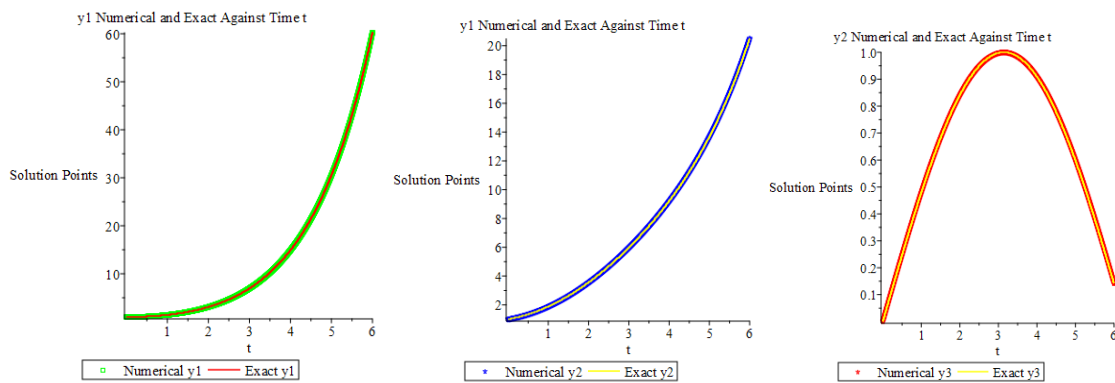
Consider the differential algebraic equations of index-1 see Moaddy et al, (2015) and Agarwal et al, (2019)

$$\left. \begin{aligned} x'(t) &= ty'(t) - t^2z(t) - x(t) + (t + 1)y(t) - (t^2 + 2t)z(t), & x(0) &= 1 \\ y'(t) &= tz'(t) + y(t) - (t - 1)z(t), & y(0) &= 1 \\ z'(t) &= \sin(t) & z(0) &= 0 \end{aligned} \right\}$$

and the exact solution is  $y(t) = e^t + t\sin(t), x(t) = e^{-t} + te^t, z(t) = \sin(t)$

**Table 1:** A comparison of absolute values between Agarwal et. al. (2019) and EHBBDF with a Wider Range.

Author's Method	t	H	Er(y <sub>1</sub> (t))	Er(y <sub>2</sub> (t))	Er(y <sub>3</sub> (t))
Hybrid Method; Agawarl et al, (2019)	3	0.001	4.78965E-08	1.53733E-08	-
	6	0.001	2.07019E-06	3.41682E-07	-
One-Leg of Hybrid Method; Agawarl et al, (2019)	3	0.0001	1.67133E-10	4.69775E-11	-
	6	0.0001	1.26784E-09	1.45462E-10	-
	3	0.001	8.87175E-06	4.70979E-06	-
	6	0.001	4.45514E-04	8.88051E-05	-
Hybrid Method; Agawarl et al, (2019)	3	0.0001	8.92249E-08	4.72626E-08	-
	6	0.0001	4.48537E-06	8.92930E-07	-
	3	0.001	1.15024E-07	3.66907E-08	-
	6	0.001	5.31317E-06	8.72459E-07	-
One-Leg of Hybrid Method; Agawarl et al, (2019)	3	0.0001	3.59840E-10	1.10736E-10	-
	6	0.0001	1.12391E-08	1.93154E-09	-
	3	0.001	1.28838E-06	9.03448E-08	-
	6	0.001	2.06502E-04	2.96481E-05	-
HBDF; Agawarl et al, (2019)	3	0.0001	1.22774E-08	1.09555E-09	-
	6	0.0001	2.02582E-06	2.90166E-07	-
	3	0.001	4.67957E-09	1.52812E-08	-
	6	0.001	7.17848E-06	1.01547E-06	-
EHBBDF	3	0.0001	2.87855E-10	9.16494E-11	-
	6	0.0001	4.06857E-08	6.55177E-09	-
	3	0.001	9.79100E-26	3.62700E-26	1.90470E-26
	6	0.001	9.59100E-25	1.96000E-25	4.08870E-26
EHBBDF	3	0.0001	6.30090E-25	1.49360E-24	1.91476E-25
	6	0.0001	2.86202E-23	1.44644e-23	4.09201e-25



**Figure 2:** A perfect and Unique Relationship between the Exact and Numerical Solution with a Wider Range of Differential States and Constraints.

Table 1, makes it evident that the recently developed method EHBBDF outperformed two schemes (HBDF and ONE-LEG) of hybrid methods developed by Agarwal et al. (2019) with a wider range in terms of the amount of global absolute error (GAE) in numerical example 3.1 while Figure 2, shows reliable, consistent and dependable relationship between the exact and numerical results from newly developed method.

**b) Numerical Example**

Consider the nonlinear semi-explicit Hessenberg DAE of index-2 type; see Ascher & Lin (1997) and Agarwal, Ibrahim, and Yousry (2019).

$$y'(t) = f(y(t), t) - B(x, t) z(t)$$

$$0 = g(y(t), t).$$

Where,

$$f = \begin{Bmatrix} 1 + \left(t - \frac{1}{2}\right) e^t \\ 2t + \left\{t^2 - \frac{1}{4}\right\} e^t \end{Bmatrix}, \quad B = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix}$$

Subject to the initial condition

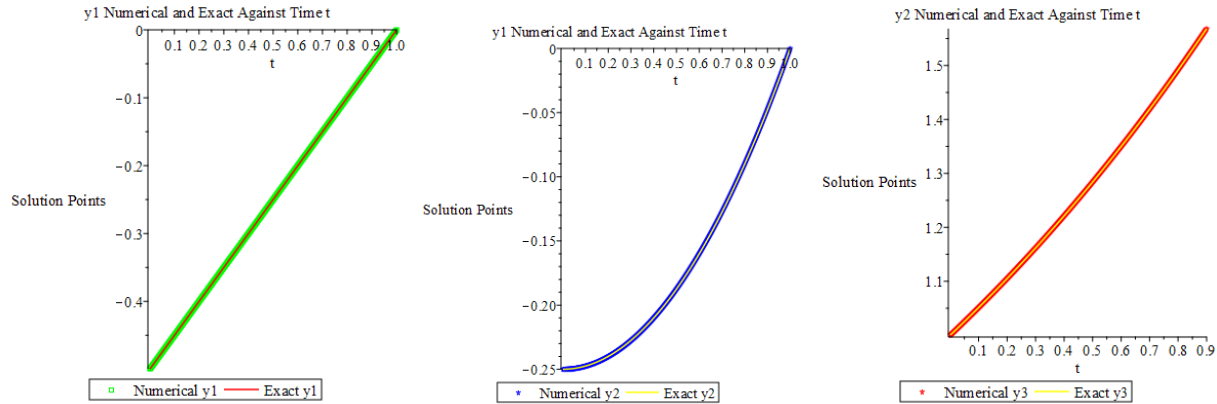
$$y_1(0) = -\frac{1}{2}, \quad y_2(0) = -\frac{1}{4}, \quad z(t) = 1.$$

The exact solution is

$$y_i(t) = \left\{t - \frac{1}{2}, t^2 - \frac{1}{4}\right\}^T, \text{ where } i = 1, 2 \text{ and } z(t) = e^t$$

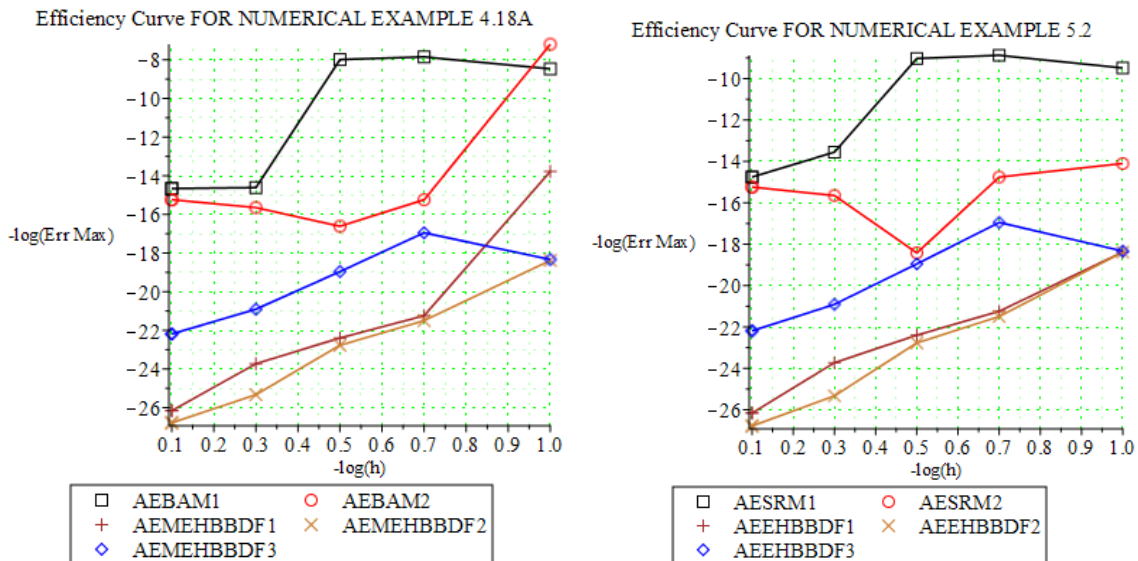
**Table 2:** Comparison of Absolute Errors  $|y(x_j) - y(x)|$  between SRM, BAMGRATE, HBDF and EHBBDF with  $h = 10^{-3}$

Authors Methods	Error @ →	t = .1	t = .3	t = .5	t = .7	t = 1.0
SRM, (1997)	Err 1	.39E-06	.13E-05	.12E-03	.14E-03	.76000E-04
	Err 2	.24E-06	.16E-06	.10E-07	.39E-06	.75000E-06
BAUMGARTE (1972)	Err 1	.43E-06	.45E-06	.34E-03	.39E-03	.21000E-03
	Err 2	.24E-06	.16E-06	.61E-07	.24E-06	.75000E-06
ONE-LEG Hybrid Method (2019)	Err 1	-	-	-	-	9.76405E-08
	Err 2	-	-	-	-	3.66007E-08
	Err 3	-	-	-	-	6.03067E-07
HBDF (2019)	Err 1	-	-	-	-	5.68471E-10
	Err 2	-	-	-	-	1.28144E-09
	Err 3	-	-	-	-	6.63956E-09
EHBBDF	Err 1	4.354E-14	4.955E-13	1.882E-12	5.948E-12	4.34500E-11
	Err 2	2.325E-14	3.003E-13	1.290E-12	4.616E-12	3.34200E-11
	Err 3	4.824E-12	2.691E-11	1.044E-10	5.473E-10	6.45300E-09



**Figure 3:** The Perfect Relationship between Exact and Numerical Solutions.

Conclusively, the graph of Figure 3 effectively demonstrates that the numerical method employed provides a very precise approximation to the exact solution for  $y_1(t)$ ,  $y_2(t)$ ,  $y_3(t)$  over the given time interval. The system modeled appears to be a stable, without any noticeable change in amplitude over time. This close agreement indicates the robustness of the numerical technique used.



**Figure 4:** Shows a comparison of the efficiency curve between newly developed methods EHBBDF, SRM, and BAMGRATE.

In a similar manner, numerical Figure 4 and Table 2 illustrate the superior accuracy of EHBBDF compared to HBDF, ONE-LEG HBDF (Agawar1 et al., 2019), SRM (Ascher & Linh, 1997), and BAUMGARTE (1972). The newly constructed methods eliminate the need for linearization, perturbation, or discretization of variables, thereby avoiding round-off errors associated with existing methods. The results indicate that the newly developed EHBBDF are effective and simple schemes characterized by discrete dynamics, making them suitable for addressing various engineering problems.

**c) Numerical Example**

We consider the following nonlinear and nonsingular Hessenberg index-2 DAE model problem (see Ascher and Lin, 1997).

$$\left. \begin{aligned} x^\mu(t) &= f(x, t) - B(x, t)y(t) \\ 0 &= g(x, t) \end{aligned} \right\}$$

where

$$f(t) = \begin{pmatrix} -x_1(t) + x_2(t) - \sin(t) - 1 - 2t \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ x_1(t) \end{pmatrix}$$

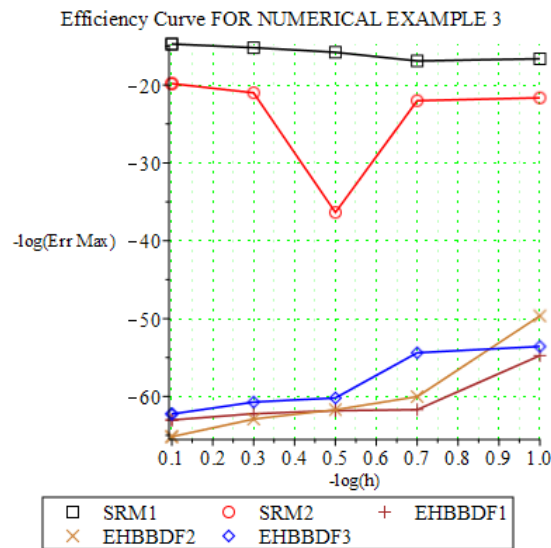
$$g(x, t) = x_1^2(t) + x_1(t)(x_2(t) - \sin(t) - 1 + 2t)$$

Subject to the initial condition  $x_1(0) = 1, x_2(0) = 0$ . The exact solution is

$$x(t) = \{(1 - 2t), \sin(t)\}^T \quad \text{and} \quad y(t) = -\frac{\cos(t)}{1 - 2t}$$

**Table 3:** Comparison of Absolute Errors  $|y(x_j) - y(x)|$  between Ascher and Lin, 1997 and EHBBDF

t	BAUMGARTE, (1972)		SRM, (1997)		EHBBDF		
	Err 1	Err 2	Err 1	Err 2	Err 1	Err 2	Err 3
0.1	$.49 \times e^{-7}$	$.39 \times e^{-7}$	$.40 \times e^{-6}$	$.25 \times e^{-8}$	$4.25 \times e^{-28}$	$4.95 \times e^{-29}$	$9.20 \times e^{-28}$
0.3	$.15 \times e^{-6}$	$.59 \times e^{-7}$	$.25 \times e^{-6}$	$.76 \times e^{-9}$	$9.58 \times e^{-28}$	$4.77 \times e^{-28}$	$4.24 \times e^{-27}$
0.5	$.93 \times e^{+1}$	$.52 \times e^{+13}$	$.14 \times e^{-6}$	$.16 \times e^{-15}$	$1.41 \times e^{-27}$	$1.59 \times e^{-27}$	$7.21 \times e^{-27}$
0.7	<i>NaN</i>	<i>NaN</i>	$.46 \times e^{-7}$	$.28 \times e^{-9}$	$1.60 \times e^{-27}$	$8.49 \times e^{-27}$	$2.38 \times e^{-24}$
1.0	<i>NaN</i>	<i>NaN</i>	$.60 \times e^{-7}$	$.40 \times e^{-9}$	$1.67 \times e^{-24}$	$2.73 \times e^{-22}$	$5.33 \times e^{-19}$



**Figure 5:** Shows a comparison of the efficiency curve between EHBBDF and SRM.

The approximate solutions of EHBBDF demonstrate accuracy, as shown in Table 3 and Figure 5, when compared to the Sequence Regularisation Method (SRM) and Baumgarte regarding global absolute errors (GAE). Table 4 demonstrates that the constructed method EHBBDF outperforms RADAU5, TSRK5, and OBRES5 regarding accuracy with broader step lengths. In Figures 5, it is evident that the recently created

methods EHBBDF outperformed SRM and BAMGRATE in Ascher and Linh (1997) in terms of accuracy level through efficiency curve.

**d) Numerical Example**

This section examines a system of nonlinear differential-algebraic equations (DAEs) of index three, as referenced in the works of Hairer and Wanner (1996, 1999), Mazzia and Magherini (2008), and Skvortsov (2012, 2015). The results obtained from solving DAE index three test problems using VDPOLE (Van der Pol’s stiff equation), PLATE (a system of ODEs derived from the discretisation of partial differential equations), and Pendulum (a system of DAEs of index three representing pendulum oscillations) are presented. The mathematical description is available in Hairer and Wanner (1996, 1999). The pendulum problem is characterised by the relationships.

$$\left. \begin{aligned} y_1'(t) &= y_3(t), & y_1(0) &= 1 \\ y_2'(t) &= y_4(t), & y_2(0) &= 0 \\ y_3'(t) &= -y_5(t)y_1(t), & y_3(0) &= 0 \\ y_4'(t) &= -y_5(t)y_2(t) - 1, & y_4(0) &= 1 \\ y_5^2(t) + y_2^2(t) &= 1, & u(0) &= 1 \end{aligned} \right\}$$

With initial values condition

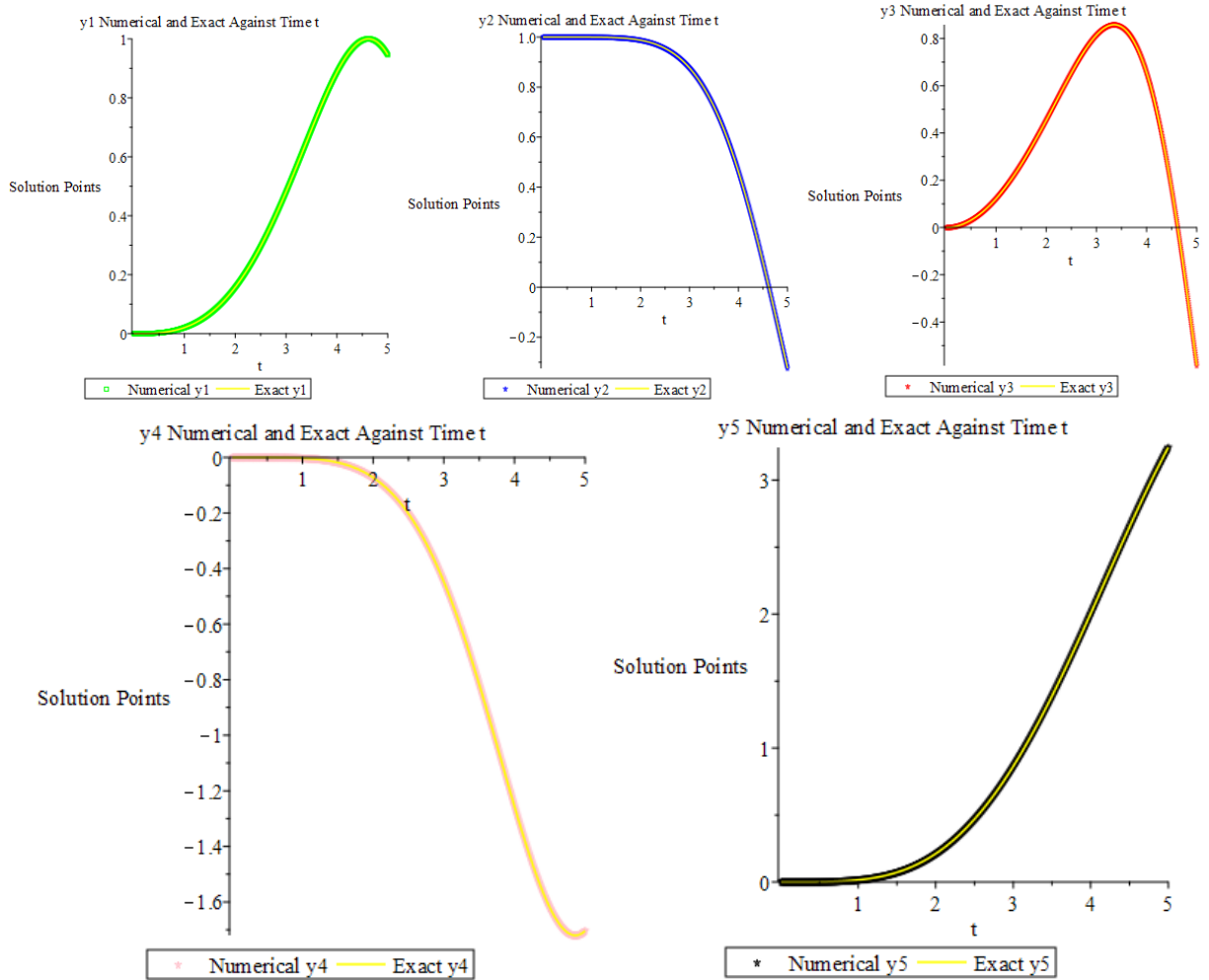
$$y_1(0) = y_3(0) = y_5(0) = y_1(0) = 0, \quad y_2(t) = 1, \quad 0 \leq t \leq 1.$$

And exact solution is

$$\begin{aligned} y_1(t) &= \sin(t - \sin t), & y_2(t) &= \cos(t - \sin t), & y_3(t) &= y_2(t)(1 - \cos t), \\ y_4(t) &= -y_1(t)(1 - \cos t), & y_5(t) &= (1 - \cos t)^2 \end{aligned}$$

**Table 4:** Shows comparison of Maximum Absolute Errors  $|y(x_j) - y(x)|$  between TSRK5, OBRES5, RADAU4 and EHBBDF when  $h = \frac{\pi}{16}$  and  $h = \frac{\pi}{160}$

Authors Methods	H	maey1	maey2	maey3	maey4	maey5
TSRK5, (2015), $\alpha = .5$	0.0625	3.65E-04	2.14E-04	8.37E-03	-	-
	0.00625	2.92E-09	1.65E-09	8.90E-07	-	-
OBRES5	0.0625	8.39E-05	5.04E-05	1.33E-04	-	-
	0.00625	9.64E-10	6.09E-10	1.48E-09	-	-
TSRK5 (2019), $\alpha = .1$	0.0625	7.43E-05	4.48E-05	1.28E-04	-	-
	0.00625	8.77E-10	5.60E-10	1.43E-09	-	-
RADAU5	0.0625	9.21E-05	2.24E-03	7.14E-02	-	-
	0.00625	9.70E-10	2.27E-06	6.94E-04	-	-
EHBBDF	0.0625	1.06500E-15	3.90690E-16	7.86563E-16	1.97841E-15	5.39186E-16
	0.00625	1.06700E-24	3.18463E-25	7.87165E-25	1.94774E-24	5.64849E-25



**Figure 6:** The Level of Comparison between Exact and Numerical Solutions Uniquely with a Wider Range

Figure 6, make clear how much more accurate the newly developed 1-step EHBBDF between exact and numerical solutions. In this case, it is shown that EHBBDF work best when the step length is very small or big, since they get more accurate as the step length decreases and increases.

#### IV. CONCLUSION

The EHBBDF method was developed, analyzed, and implemented using MAPLE, 2016, to address nonlinear coefficient variables of semi-explicit Hessenberg differential-algebraic equations (SEHDAEs) of various indexes by transforming SEHDAEs into ordinary differential equation systems through differentiation arrays. The methods integrated the primary and supplementary approaches to address semi-explicit Hessenberg differential algebraic equations related to the problem. The numerical method demonstrates greater accuracy, consistency, convergence, and efficiency compared to the existing method, highlighting its potential for addressing coefficient variables in semi-explicit Hessenberg differential-algebraic equations (SEHDAEs).

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