

ALGEBRAIC STRUCTURE OF PERFECT-PRESERVING FULL TRANSFORMATION SEMIGROUP

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ABSTRACT

The full transformation semigroup T_n is a fundamental object in finite semigroup theory, and imposing restrictions on the fibres of its transformations gives rise to important structural subclasses. This paper investigates the algebraic structure of the perfect-preserving full transformation semigroup Pf_n , a subclass of the full transformation semigroup T_n consisting of all transformations $f: X_n \rightarrow X_n$ where $X_n = \{1, 2, \dots, n\}$, satisfying the condition: $|f^{-1}(y)| \geq 2$ for every y in the image of f . The study aims to characterize the fundamental algebraic properties of Pf_n including its rank structure, idempotents, regularity, ideals, and Green's relations. An algebraic approach based on kernel block/partitions, image sets, rank, and function composition is employed to study the elements of Pf_n . The regularity of the semigroup is established by constructing a perfect-preserving pseudo-inverse for an arbitrary element, while kernel and image classifications are used to investigate its Green classes and rank-based ideal structure. The results show that every element of Pf_n is singular and has rank at most $\lfloor n/2 \rfloor$, while non-constant perfect-preserving transformations exist if and only if $n \geq 4$. The idempotents in Pf_n are characterized as those idempotent transformations for which every fixed point has at least one additional preimage. Furthermore, its ideals are organised according to admissible ranks, with the constant transformations forming the minimal ideal, and its Green-class decomposition is described in terms of kernel and image structures. These results provide a foundation for further combinatorial and algebraic investigations of perfect-preserving transformation semigroups.

KEYWORDS: Full Transformation Semigroup, Perfect-Preserving Transformation, Pseudo-Inverse, Kernel, Ideals, Regular Semigroup, Idempotent, Green's Relations, Singular Transformations.

I. INTRODUCTION

The formalization of semigroup theory as a distinct mathematical discipline occurred primarily during the mid-twentieth century, evolving from a need to describe structures that do not necessarily satisfy the rigid requirements of group theory. A semigroup $(G, \cdot)$ is defined fundamentally as an arbitrary set G combined with an associative binary operation, where

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all elements in the set (Pichor and Rudnicki, 2024). While group theory focuses on symmetries and reversible operations, semigroups are better suited for modelling irreversible transformations, making them "transformation semigroups everywhere" in mathematical modelling (Pichor and Rudnicki, 2024). When these elements act as transformations on a space X under function composition, they form the cornerstone of modern algebraic investigation (Pichor and Rudnicki, 2024).

The full transformation semigroup on a set X , frequently denoted as T_X or T_n , serves as the universal object in this field (Sarkar and Singh, 2020). Its significance is comparable to that of the symmetric group S_X in group theory, as T_X contains all possible mappings from a set into itself (Sarkar and Singh, 2020). Much of the research in this area explores subsemigroups of

T_n that satisfy specific constraints, such as those preserving a binary relation, a fixed set, or a specific partition (Sarkar and Singh, 2020,2022). For instance, scholars have investigated semigroups with restricted partial ranges, denoted $T(X,Y)$, where the image of every transformation must fall within a specific subset Y of X (Jin and You, 2023). These investigations often seek to determine fundamental algebraic features, such as the number of elements (cardinality), the structure of generating sets, and the nature of automorphisms (Ayik and Bugay, 2014; Jin and You, 2023). The study of structural properties within transformation semigroup has been an active area of research. For instance, (Mogbonju, Oguleke and Adeniji, 2019) investigated polarity within the signed transformation semigroup, while more recent works have explored topological structures (Francis, Adeniji, and Mogbonju, 2023) and conjugacy classes within these semigroups (Peter, Mogbonju, and Adeniji, 2024). This research continues this line of inquiry by introducing and analysing a new class: the perfect-preserving full transformation semigroup.

For a transformation $f \in T_n$, the kernel blocks $f^{-1}(y)$, for $y \in Im(f)$, form the nonempty blocks of the kernel partition of f . In an unrestricted transformation, a kernel block may be a singleton. This paper investigates the effect of excluding singleton kernel blocks and the resulting class of transformation is termed Perfect-preserving full transformation semigroup. Thus we have its definition as follows:

A transformation $f \in T_n$ is said to be perfect preserving if every image element has at least two preimages and no singleton kernel classes. Equivalently, Let $f : X_n \rightarrow X_n$ where $X_n = \{1,2,\dots,n\}$ and $f \in T_n$ be the full transformation semigroup. We say f is a perfect-preserving transformation if $|f^{-1}(y)| \geq 2$ for every y in the image of f .

The term “perfect” is chosen to reflect the “complete” or “balanced” coverage of the image set, where no element is a singleton preimage. The term “preserving” is used because this preimage cardinality property is closed under the operation of composition, making it a natural algebraic invariant of the semigroup. Hence the term “Perfect-preserving transformation” is introduced to emphasize that every image point is associated with a non-trivial kernel class. Unlike arbitrary transformations, where singleton kernel classes may occur, each image is supported by a non-singleton fibre, preserving the collective structure of the domain through non-trivial kernel partitions.

We define a transformation to be perfect-preserving if $|f^{-1}(y)| \geq 2 \forall y \in Im(f)$. The associated class is denoted by Pf_n . The terminology reflects the requirement that every image point be supported by a nontrivial kernel block. The minimum kernel block condition has immediate combinatorial consequences. If $r = |Im(f)|$, then the kernel blocks partition an n -element set into r blocks, each of size at least two. Consequently, $2r \leq n$. Thus the admissible ranks are bounded by $\lfloor n/2 \rfloor$, and the symmetric group does not occur in Pf_n . At the same time, the class is sufficiently rich to contain constant maps, non-constant maps when $n \geq 4$, idempotents, and regular elements.

II. RELATED WORK

A. The Full Transformation Semigroup and its Principal Subclasses

The full transformation semigroup, denoted as T_n , stands as one of the most essential structures in modern semigroup theory. Its central importance is derived from its universal property, which serves as an analogue of Cayley’s theorem for groups: every abstract semigroup can be embedded as a subsemigroup within a full transformation semigroup on some set (Dolinka and East, 2015). The theoretical foundations of T_n were significantly expanded during the mid-twentieth century. A landmark contribution was made by John Howie in 1966, who proved that the set of singular transformations (those that are not permutations) is generated by its

idempotents (Howie, 1966). This discovery catalysed a wide range of studies into the generative properties of transformation monoids. Researchers have since sought to determine the rank and idempotent rank of various sub-semigroups to understand their minimal structural requirements (Dolinka, East and Mitchell, 2015). This foundational characteristic positions T_n as a primary vehicle for exploring more specialized algebraic structures that are defined by specific functional constraints like the one defined by the class Pf_n . For $\alpha \in T_n$, the image, kernel and rank are defined by:

$$\text{Im}(\alpha) = \{x\alpha : x \in X\}, \text{ker}(\alpha) = \{(x, y) : \alpha x = \alpha y\}, \text{and } \text{rank}(\alpha) = |\text{Im}(\alpha)|. \\ |Sing_n| = n^n - n! \quad (2.1)$$

The rank of a transformation is also its D-class parameter in the full transformation semigroup. More specifically, two elements of T_n are R-related when they have the same image, L-related when they have the same kernel, and D-related when they have the same rank. Thus, the rank stratification of T_n is one of the basic structural facts on which the study of restricted transformation semigroups is built (Howie, 1976).

Several important subclasses arise by restricting the domain, image, kernel or action of a transformation. The partial transformation semigroup PT_n contains maps whose domains may be proper subsets of X_n . The partial one-one transformation semigroup contains the injective partial maps and is an inverse semigroup. The order-preserving transformation semigroup consists of maps satisfying; $i < j \Rightarrow \alpha i \leq \alpha j$, whereas an order-preserving injective map satisfies the strict inequality $\alpha i < \alpha j$. These classes should not be conflated: order preservation is an order condition, injectivity is a fibre condition, and partiality is a domain condition. Other classes restrict the range or the kernel. Given $A \subseteq X$, the restricted-range semigroup is;

$$T(X, A) = \{\alpha \in T_X : \text{Im}(\alpha) \subseteq A\} \quad (1)$$

Given an equivalence relation ρ on X , a kernel-preserving or partition-preserving class may require $\text{ker}(\alpha)$ to contain, or interact compatibly with ρ . (Dolinka and East, 2015) showed that restricted-range and restricted-kernel semigroups arise naturally in the study of variants of T_X . Their work is useful here because it demonstrates how a local condition on images or kernels changes the regular elements, Green's relations, ideals and generating theory of a transformation semigroup.

B. Regularity and Idempotents in Transformation Semigroups

An element a of a semigroup S is regular if there exists $b \in S$ such that $aba = a$. A semigroup is regular if all of its elements are regular. The full transformation semigroup T_n is regular. For a transformation ($\alpha \in T_n$), one constructs an inverse-like transformation by selecting one representative from each fibre and mapping the image back to the selected representatives. This representative map need not be unique, so it is a von Neumann inverse rather than necessarily a group inverse. The restriction to a subsemigroup is significant. A transformation may be regular in T_n while none of its chosen inverses belongs to the restricted subsemigroup. Consequently, regularity must be proved relative to the class under study. This point is directly relevant to the perfect-preserving semigroup: a pseudo-inverse must preserve the condition that every image point has at least two preimages. An element e is idempotent if $e^2 = e$. In a transformation semigroup, every point of the image of an idempotent is fixed. Conversely, a transformation whose image consists of fixed points is idempotent. Therefore, an idempotent is determined by its image together with the assignment of the remaining domain points to those fixed image points. Howie's foundational work established that the singular transformation semigroup is generated by its idempotents (Howie, 1966). The result identifies idempotents as sufficient to generate every non-permutation transformation. Howie later studied minimal idempotent generating sets. In the 1978 paper *Idempotent generators in finite*

full transformation semigroups, he stated that for $n \geq 3$, a minimal idempotent generating set for the singular part has $\frac{1}{2}n(n-1)$ elements, and he gave a formula for the number of such minimal sets (Howie, 1978). These results explain why idempotents and their ranks are central in the study of singular and restricted transformation semigroups. (Zhao, 2012) classified maximal idempotent-generated sub-semigroups of finite orientation-preserving singular partial transformation semigroups. Zhao's work shows that the question is not only whether a semigroup contains idempotents, but also which sub-semigroups they generate and how the order restriction changes the answer. These studies provide useful comparisons for the characterization of idempotents in Pf_n , but their conditions are different from the minimum-fibre condition used here.

C. Green's relations, ideals and rank structure

Green's relations describe the principal ideal structure of a semigroup. For $a, b \in S$, the relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ compare the left ideals, right ideals, two-sided ideals, intersections of left and right classes, and the combined structure generated by these ideals. In T_n , the relations have a concrete mapping interpretation: $\mathcal{L}$ is determined by the kernel, $\mathcal{R}$ by the image, $\mathcal{D} = \mathcal{J}$ and by rank. The two-sided ideals of T_n are consequently rank ideals of the form;

$$I_r = \{\alpha \in T_n : \text{rank}(\alpha) \leq r\} \quad (2)$$

For a proper subsemigroup, these descriptions may no longer hold without modification. A restricted class can contain only some transformations with a given kernel, image or rank. (Dolinka and East, 2015) investigated variants T_X^a , whose multiplication is defined by $x * y = xay$. They described the regular part of the variant, characterized its Green's relations, examined its ideals, and calculated ranks and idempotent ranks in the relevant cases. Their results show how a change in the multiplication or an image/kernel restriction can preserve the broad rank-based picture while changing the number and arrangement of individual Green classes. Green's relations are also used in recent work on transformation semigroups defined by geometric or combinatorial restrictions. (Ahmadu, Mogbonju and Adeniji, 2025) studied a star-like full transformation semigroup. They introduced height, fixed points, collapse and relapse parameters, and characterized the $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ relations for the star-like class. Their work is especially relevant to the present dissertation because it combines explicit enumeration with a Green-relation analysis of a proper transformation subsemigroup. The defining star-like operator, however, is geometric and not a minimum-fibre condition.

D. Order-Preserving, Partition-Preserving and Partial Transformations

Order-preserving semigroups form one of the most studied families of restricted transformation semigroups. Their elements respect a prescribed linear order, so their images and kernel blocks must occur in an ordered pattern. This order restriction greatly reduces the number of possible transformations and changes the structure of idempotent-generated sub-semigroups. (Zhao, 2011) studied idempotent rank in decreasing and order-preserving partial transformation semigroups, obtaining exact generation results for the relevant finite classes. (Zhao, 2012) further considered orientation-preserving singular partial transformations and classified maximal idempotent-generated sub-semigroups. Conjugacy provides another way to classify restricted transformations. (Mogbonju, Gwary and Ojeniyi, 2014) presented conjugacy classes in a partial order-preserving transformation semigroup with the aid of unlabelled graphs. Their approach arranged elements according to graph representations rather than only rank or kernel type. This is relevant to the kernel-type conjugacy considered in the present dissertation, although graph-based conjugacy in an order-preserving partial class is not the same as

conjugacy in the full minimum-fibre class. Partition-preserving transformation semigroups impose a different type of restriction. Instead of controlling the cardinality of each fibre, they require transformations to preserve a fixed partition or equivalence relation. (Sarkar and Singh, 2020) studied regular, unit-regular and idempotent elements in transformation semigroups that preserve a partition. Their results show that the regularity and idempotent structure depend on how the transformation acts on the blocks of the partition. This comparison is important: both partition preservation and perfect-preservation constrain kernels, but partition preservation refers to a prescribed external equivalence relation, whereas perfect preservation imposes a lower bound on every non-empty kernel block.

E. Fibre-size Restrictions and the Perfect-Preserving Class

The perfect-preserving full transformation semigroup studied in this work is defined as:

$$Pf_n = \{ \alpha \in T_n : |\alpha^{-1}(y)| \geq 2 \ \forall y \in Im(\alpha) \}$$

Equivalently, every block of $\ker(\alpha)$ has size at least two. The terminology used here is specific to the study and should not be confused with the permutation property P_n , which concerns reordering of products in a semigroup. The two notions concern different mathematical objects: P_n is a law involving products of semigroup elements, while Pf_n is a condition on the fibres of one transformation.

The minimum-fibre condition has immediate consequences. If $\alpha \in Pf_n$ has rank r , then the r non-empty fibres partition X_n , and each has at least two elements. Hence $n \geq 2r$ and therefore, $\text{rank}(\alpha) = r \leq \lfloor n/2 \rfloor$. No element of Pf_n is injective when $n \geq 1$, so $Pf_n \cap S_n = \emptyset$. Consequently, $Pf_n \subset Sing_n$. This makes the Pf_n a rank-truncated singular subsemigroup, but it is not the same as the full singular semigroup: the latter allows singleton kernel blocks, whereas former excludes them. Related research has considered an upper bound on fibre sizes rather than the lower bound used here. For instance, recent studies have linked transformation semigroups to graph theory. (Margolis and Rhodes, 2021,2023) studied transformation monoids of degree 2, where the inverse image of any point contains at most two elements. They were able to determine the Krohn-Rhodes complexity of this transformation monoid of degree 2. Margolis and Rhodes found that the complexity of such actions is at most 2, demonstrating a clear connection between fibre size constraints and global algebraic complexity. This suggests that semigroups like Pf_n , which impose similar fibre-size restrictions, may possess unique structural properties that simplify their classification compared to the full T_n monoid. An upper bound and a lower bound produce different classes and should not be treated as equivalent. The degree-two results therefore provide conceptual background about the effect of fibre restrictions, but they do not establish the algebraic properties of Pf_n . (Mogbonju, Adeniji and Ogunleke, 2019) studied signed singular mapping transformation semigroups. They defined signed transformations with images in an enlarged signed set, investigated the order and cardinality of signed singular mappings, and examined idempotent, nilpotent and chain-decomposition properties. Their paper also records the classical cardinality and idempotent formulas for full, partial and singular transformation semigroups. The signed setting differs from Pf_n , but the treatment of singularity, idempotents and enumeration is directly relevant. (Mogbonju *et al.*, 2019) studied polarity in the signed symmetric group and signed transformation semigroup. Their work examined polarity as a structural property of signed mappings and related it to signed singular transformations. This provides an example of how a transformation semigroup can be refined by an additional combinatorial operation rather than by a fibre restriction. (Francis *et al.*, 2023) introduced m -topological transformation semigroup regular spaces. They defined a work-done function based on the distance $|x - \alpha x|$, and derived formulas for the total work, average work and power associated with the

transformations. Their results show how transformation semigroups can be studied simultaneously through algebraic structure and an external geometric or topological model. The present study is algebraic and combinatorial, but it shares the aim of extracting explicit formulas from a restricted transformation class. (Peter *et al.*, 2024) studied a star-like three-dimensional conjugacy class on partial one-one transformation semigroups. They introduced star-like geometric parameters, including derank, collapse, relapse, pivot and joint, and concluded that the class has n order conjugacy classes under their definitions. This work is relevant to the dissertation's conjugacy objective because it demonstrates a recent application of geometric parameters to conjugacy in a restricted transformation semigroup. Its partial one-one hypothesis and star-like geometry are, however, distinct from the no-singleton-fibre condition defining Pf_n .

III. METHODOLOGY AND DISCUSSION

A. Formal Structural Analysis of Perfect-preserving full transformation semigroup, Pf_n .

The investigation was initiated by analysing the defining mapping constraints of the perfect-preserving full transformation semigroup denoted as Pf_n . As a substructure of the full transformation semigroup T_n , Pf_n was defined as the set of transformations where each element in the image had a preimage cardinality of at least two. This restriction on the kernel block sizes led to several immediate and critical structural observations regarding the nature of these mappings. The analysis focused on how these local kernel block constraints influenced the global algebraic properties of the semigroup, particularly concerning injectivity, rank, and kernel decomposition.

(a). Categorical Exclusion of Injective Mappings

The first significant observation concerns the relationship between Pf_n and the symmetric group S_n . By definition, an injective map requires that every element in the domain is mapped to a unique element in the image, resulting in kernel blocks of size exactly one. Since the core requirement of Pf_n was that every image point must possess at least two preimages, all injective maps were fundamentally excluded from the set. This led to the formal conclusion that the intersection of the symmetric group S_n and Pf_n is the empty set. i.e., $S_n \cap Pf_n = \emptyset$. Consequently, this exclusion highlights that Pf_n is composed entirely of singular transformations, aligning with the broader classification of singular sub-semigroups.

(b). Constraints on Rank and Hierarchical Layering

This research further explores the dimensional limitations of transformations within Pf_n . If $f \in Pf_n$ and r represents the rank (the size of the image), the total size of the domain n must satisfy the inequality $n \geq 2r$. This conclusion is derived from the fact that each of the r image points was associated with at least two distinct elements in the domain. Consequently, the rank of any element in Pf_n was bounded such that $r \leq \lfloor n/2 \rfloor$. This finding is significant because it positions the elements of Pf_n exclusively within the lower-half rank layers of the full transformation semigroup T_n . This hierarchical restriction distinguished Pf_n from other subsemigroups that spanned the full range of possible ranks.

(c). Characterization of Kernel Structures

The kernel of a transformation, defined as the equivalence relation that partitions the domain into kernel blocks. The kernel of a perfect preserving transformation determines the internal structure of Pf_n elements. A transformation belongs to Pf_n if and only if its kernel

decomposition contained no singleton blocks. Thus in Pf_n , every kernel block has size greater than or equal to two. Consequently, a perfect preserving transformation semigroup can also be referred to as a perfect-kernel decomposition of full transformation semigroup since the size of each kernel must be greater than or equal to 2.

B. Enumeration of the Elements of Pf_n .

Perfect preserving transformation is defined as follows:

$$Pf_n = \{ f \in T_n : |f^{-1}(y)| \geq 2 \ \forall y \in Im(f) \} \quad (3)$$

To validate the theoretical observations about Pf_n , a systematic enumeration of its elements is conducted for small values of n . The method of two-line notation is employed to provide a clear and verifiable representation of each transformation.

For $n = 1$, there is no perfect preserving transformation. i.e.

$$Pf_1 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right) \notin Pf_n$$

Thus $Pf_1 = \emptyset$

Therefore, $|Pf_1| = 0$

In Pf_1 , perfect-preserving transformation cannot exist since the single element in the domain cannot satisfy the requirement of having at least two preimages, resulting in $|Pf_1| = 0$.

As the value of n increases, the transformations are identified base on the rank bounds established in the previous section 3.2.2.

The following are perfect preserving transformations in Pf_2

For $n=2$, $r \leq \lfloor n/2 \rfloor$ implies that $r \leq \lfloor \frac{2}{2} \rfloor \leq 1$.

Thus, the following are the perfect preserving transformations in Pf_2

$$Pf_2 = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix} \right) \right\}$$

$$|Pf_2| = 2$$

In Pf_2 , only 2 perfect preserving transformations both of rank 1 exist, leading to the identification of two specific transformations; where both elements in the domain are mapped to the same image point.

For $n = 3$, $r \leq \lfloor n/2 \rfloor$ implies that $r \leq \lfloor \frac{3}{2} \rfloor \leq 1$.

Thus, the following are perfect preserving transformations in Pf_3

$$Pf_3 = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{smallmatrix} \right) \right\}$$

$$|Pf_3| = 3$$

In Pf_3 , the rank remains constrained to 1, and the total count of valid transformations is found to be three. This process continues for $n = 4$, where the rank could be either 1 or 2, significantly increasing the complexity and quantity of the elements.

For $n = 4$: $r = \lfloor \frac{4}{2} \rfloor = 2$.

For $n = 4$, $r \leq \lfloor n/2 \rfloor$ implies that $r \leq \lfloor \frac{4}{2} \rfloor \leq 2$. So $r=1$ or $r=2$

Thus there are four constant maps, each of rank 1.

$$Pf_4 = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{smallmatrix} \right) \right\}$$

Next, we list out all the rank 2 elements of Pf_4 . The kernel must have exactly 2 blocks to be a rank 2 element of Pf_4 , and each block must have size 2. So, we first list all partitions of the set $\{1,2,3,4\}$ into 2-element blocks. Thus, there are exactly 3 such partitions. $\{\{1,2\}, \{3,4\}\}$, $\{\{1,3\}, \{2,4\}\}$, and $\{\{1,4\}, \{2,3\}\}$.

Partition $\{\{1,2\}, \{3,4\}\}$.

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 3 \end{pmatrix} \right\}$$

Partition $\{\{1,3\}, \{2,4\}\}$.

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{pmatrix} \right\}$$

Partition $\{\{1,4\}, \{2,3\}\}$.

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 3 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \right\}$$

There are 12 transformations in each partition. Thus, there are 36 rank 2 elements in Pf_4 .

Therefore, there is a total of 40 elements in Pf_4 , comprising 4 constant maps of rank 1 and 36 non-constant maps of rank 2.

Hence, $|Pf_4| = 40$ perfect-preserving transformations.

Proposition 1

$(Pf_n, \circ)$ is a subsemigroup of $(T_n, \circ)$. That is, the elements of Pf_n are closed under the operation of composition, thus Pf_n is a subsemigroup of the full transformation semigroup T_n .

Proof:

Let $f, g \in Pf_n$. We must show that the composition $f \circ g \in Pf_n$. Take any element $z \in Im(f \circ g)$. We need to show that $|(f \circ g)^{-1}(z)| \geq 2$. Since $z \in Im(f \circ g)$, there exists some $x \in X_n$ such that $z = f(g(x))$. This implies that $z \in Im(f)$ and $g(x) \in f^{-1}(z)$. Because $f \in Pf_n$, the kernel block of z under f has size at least two. i.e $|f^{-1}(z)| \geq 2$. Now consider the kernel block of z under the composition $f \circ g$:

$$(f \circ g)^{-1}(z) = \{x \in X_n : f(g(x)) = z\} \quad (4)$$

This set is exactly the union of the kernel blocks of g over all $y \in f^{-1}(z)$.

$$(f \circ g)^{-1}(z) = \bigcup_{y \in f^{-1}(z)} g^{-1}(y) \quad (5)$$

Since g is a function, the set $g^{-1}(y)$ for distinct y are pairwise disjoint. Thus, the cardinality of the union is the sum of cardinalities;

$$|(f \circ g)^{-1}(z)| = \sum_{y \in f^{-1}(z)} |g^{-1}(y)| \quad (6)$$

Now, because f is perfect-preserving, the kernel block over z has size at least, i.e $|f^{-1}(z)| \geq 2$. Let these preimage be $y_1, y_2, \dots, y_m$ where $m = |f^{-1}(z)| \geq 2$. Because g is also perfect-preserving, for each of these y_i , we have $|g^{-1}(y_i)| \geq 2$ for all $i = 1, 2, \dots, m$.

Because $f^{-1}(y)$ contains at least two elements (say y_1 and y_2), and because $g \in Pf_n$, we know that for every $y_i \in f^{-1}(y) \subseteq Im(g)$. We have: $|g^{-1}(y_i)| \geq 2$

Therefore, the sum is at least;

$$|(f \circ g)^{-1}(z)| = \sum_{i=1}^m |g^{-1}(y_i)| \geq \sum_{i=1}^m 2 \geq 2m \geq 4 \geq 2 \quad (7)$$

Since $z \in \text{Im}(f \circ g)$ was arbitrary, every element in the image of $f \circ g$ has at least two preimages. Hence $f \circ g \in Pf_n$. Thus Pf_n is closed under composition.

Composition of functions is associative in T_n . That is, for any $f, g, h \in T_n$,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Since $Pf_n \subseteq T_n$, this associativity property is inherited directly. Thus, for any $f, g, h \in Pf_n$,

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Since Pf_n is nonempty class that is closed and associative under the operation of composition (inherited from T_n), it follows that $(Pf_n, \circ)$ is a subsemigroup of $(T_n, \circ)$ and hence a semigroup in its own right. $\square$

Below is an illustrative example for clarity of our proposition.

Let $f, g \in Pf_4$ such that $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}$.

$f^{-1}(1) = \{1, 2\}$, which implies that $|f^{-1}(1)| \geq 2$

$f^{-1}(2) = \{3, 4\}$, which implies that $|f^{-1}(2)| \geq 2$

$g^{-1}(1) = \{1, 3\}$, which implies that $|g^{-1}(1)| \geq 2$

$g^{-1}(2) = \{2, 4\}$, which implies that $|g^{-1}(2)| \geq 2$

Let $h = f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

This implies that; $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$h^{-1}(1) = \{1, 2, 3, 4\}$.

$|h^{-1}(1)| = 4 \geq 2$.

Therefore, $h = f \circ g \in Pf_4$

Proposition 2 (Existence of non-constant transformations)

Non-constant perfect transformations exist in T_n if and only if $n \geq 4$.

Proof:

Rank of a non-constant map must satisfy; $r \geq 2$. But perfectness requires that $2r \leq n$. If $r = 2$, then $4 \leq n$. Hence $n \geq 4$ as required. $\square$

Proposition 3 (Enumeration formula for the Cardinality of Pf_n)

For $n \geq 2$, the number of perfect-preserving elements in T_n for an n -element set is:

$$|Pf_n| = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} r! S_{n \geq 2}(n, r) \quad (8)$$

Proof:

Let $f \in Pf_n$ have rank r . Then $|\text{Im}(f)| = r$ and $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$ because each of the r image points require at least 2 preimages, so $n \geq 2r$. To construct such a transformation, we perform three independent choices. First, partition the domain into fibres. The kernel of f partitions the n -element domain $X_n = \{1, 2, \dots, n\}$ into exactly r unlabelled blocks. These blocks are precisely the fibres $f^{-1}(y)$ for each image point $y \in \text{Im}(f)$. Because f is perfect preserving, every block must have size at least 2. The number of ways to partition an n -set into r unlabelled blocks, each of size at least 2 is by definition the restricted Stirling number $S_{n \geq 2}(n, r)$. Secondly, we choose the image set. Thus we now decide which r distinct elements of X_n will be the image of f . This is simply choosing a subset of size r from n element which is by definition $\binom{n}{r}$. Lastly, we assign the fibre to each image. Since we have r unlabelled fibres and r labelled image point, we assign each fibre to exactly one image point such that every image point must receive exactly one fibre. This is by definition a bijection between r fibres and r

image points, giving the number of ways to make such assignments $r!$ ways. Thus, for a fixed rank r , the number of perfect-preserving transformations of rank r is: $\binom{n}{r} r! S_{n \geq 2}(n, r)$. Finally, since r can range from 1 up to $\lfloor \frac{n}{2} \rfloor$, we sum over all possible ranks, which yields the formula.

Computation of the Cardinalities of Pf_n for $n = 2, 3, \dots, 10$

From proposition 3, above, it has been established that cardinality of Pf_n is obtained as: $|Pf_n| = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} r! S_{n \geq 2}(n, r)$, where $S_{n \geq 2}(n, r)$ are the associated Stirling numbers of the second kind counting partitions of n -set into r blocks each of size at least 2. These numbers have known tables and recurrence relations and are applied to compute the cardinalities of Pf_n as shown in Table 1.

C. Idempotents in Pf_n with their Cardinalities.

An idempotent element in a semigroup is an element g such that $g^2 = g$. The above definition also holds for idempotents in Pf_n because every element fixes its image set. Idempotents in Pf_n are characterized by the following:

- image set: $Y = Im(f) \subseteq [n]$ with $|Y| = r \leq \lfloor \frac{n}{2} \rfloor$ where r is the rank of f ;
- fixed points: every y in Y satisfies $f(y) = y$;
- preimage condition: for each y in Y , the set $f^{-1}(y)$ has size at least 2. (so y itself plus at least one other element); and
- partition structure: the set $\{f^{-1}(y) : y \in Y\}$ forms a partition of $[n]$ into r blocks, each containing its image and at least one other element.

The following are idempotent perfect preserving transformations in Pf_2

$$E(Pf_2) = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \right\}$$

$$|E(Pf_2)| = 2$$

The following are idempotent perfect preserving transformations in Pf_3

$$Pf_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \right\}$$

$$|E(Pf_3)| = 3$$

$$E(Pf_4) =$$

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \right\}$$

$$|E(Pf_4)| = 16$$

Note: all the constant transformations in Pf_n are idempotents.

All the other elements that do not fix their image sets are non-idempotent:

Table 1: Computations for the cardinality of Pf_n based on rank using Stirling numbers of the second kind.

n	r	$\binom{n}{r} r!$	$S_{n \geq 2}(n, r)$	$\binom{n}{r} S_{n \geq 2}(n, r)$	$\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} r! S_{n \geq 2}(n, r)$
2	1	$\frac{2!}{1!} = 2$	$S_{n \geq 2}(2,1) = 1$	2	2
3	1	$\frac{3!}{2!} = 3$	$S_{n \geq 2}(3,1) = 1$	3	3
4	1	$\frac{4!}{3!} = 4$	$S_{n \geq 2}(4,1) = 1$	4	
4	2	$\frac{4!}{2!} = 12$	$S_{n \geq 2}(4,2) = 3$	36	40
5	1	$\frac{5!}{4!} = 5$	$S_{n \geq 2}(5,1) = 1$	5	
5	2	$\frac{5!}{3!} = 20$	$S_{n \geq 2}(5,2) = 10$	200	205
6	1	$\frac{6!}{5!} = 6$	$S_{n \geq 2}(6,1) = 1$	6	
6	2	$\frac{6!}{4!} = 30$	$S_{n \geq 2}(6,2) = 25$	750	
6	3	$\frac{6!}{3!} = 120$	$S_{n \geq 2}(6,3) = 15$	1800	2,556
7	1	$\frac{7!}{6!} = 7$	$S_{n \geq 2}(7,1) = 1$	7	
7	2	$\frac{7!}{5!} = 42$	$S_{n \geq 2}(7,2) = 56$	3252	
7	3	$\frac{7!}{4!} = 210$	$S_{n \geq 2}(7,3) = 105$	22,050	24,409
8	1	$\frac{8!}{7!} = 8$	$S_{n \geq 2}(8,1) = 1$	8	
8	2	$\frac{8!}{6!} = 56$	$S_{n \geq 2}(8,2) = 119$	6664	
8	3	$\frac{8!}{5!} = 336$	$S_{n \geq 2}(8,3) = 490$	164,640	
8	4	$\frac{8!}{4!} = 1680$	$S_{n \geq 2}(8,4) = 105$	176,400	347,712
9	1	$\frac{9!}{8!} = 9$	$S_{n \geq 2}(9,1) = 1$	9	
9	2	$\frac{9!}{7!} = 72$	$S_{n \geq 2}(9,2) = 246$	17,712	
9	3	$\frac{9!}{6!} = 504$	$S_{n \geq 2}(9,3) = 1918$	966,672	
9	4	$\frac{9!}{5!} = 3024$	$S_{n \geq 2}(9,4) = 1260$	3,810,240	4,794,633

10	1	$\frac{10!}{9!} = 10$	$S_{n \geq 2}(10,1) = 1$	10	
10	2	$\frac{10!}{8!} = 90$	$S_{n \geq 2}(10,2)=501$	45,090	
10	3	$\frac{10!}{7!} = 720$	$S_{n \geq 2}(10,3)=6825$	4,914,000	
10	4	$\frac{10!}{6!} = 5040$	$S_{n \geq 2}(10,4)=9450$	47,628,000	
10	5	$\frac{10!}{5!} = 30240$	$S_{n \geq 2}(10,5) = 945$	28,576,800	81,163,900

Proposition 4

The idempotent elements of Pf_n are precisely the idempotent transformations on X_n for which every fixed point has at least one distinct element mapped to it.

Proof:

Let $f \in Pf_n$ be idempotent. Since $f^2 = f$, then every image point of f is a fixed point. Therefore $Im(f) = Fix(f)$, where $Fix(f)$ denotes the set of fixed points of f . By the definition of Pf_n , every image point has at least two preimages. Hence every fixed point belongs to a kernel block containing itself and at least one additional element. Conversely, suppose f is idempotent and every fixed point has at least one distinct preimage. Since every image point of an idempotent is a fixed point, then every image has at least two preimages. Thus $f \in Pf_n$. Hence, the two classes coincide. $\square$

Proposition 5 (Enumeration formula for the Cardinality of idempotents in Pf_n ($E(Pf_n)$))

The number of perfect-preserving idempotent elements in T_n , is:

$$|E(Pf_n)| = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} r! S(n-r, r) \quad (9)$$

Proof:

To count idempotents of rank r in Pf_n , choose the image set Y of size r , $\binom{n}{r}$ ways.

The remaining $(n-r)$ elements (non-fixed points) must be distributed among the r image elements, with each image element receiving at least one preimage (to ensure size ≥ 2). This is equivalent to counting surjections from $(n-r)$ -element set to r -element set, where the r image elements are labelled by the elements of Y . The number of surjections from an m -element set onto an r -element set is $r! S(m, r)$, where $S(m, r)$ is the associated Stirling number of the second kind. Here, $m = (n - r)$ are the non-fixed point and we need to assign them to the r labelled image elements, such that each element gets at least one. Thus the number of idempotents of rank r is; $\binom{n}{r} r! S(n-r, r)$. Summing over all possible ranks from the ranks from $r = 1$ up to $\lfloor \frac{n}{2} \rfloor$ yields the formula. $\square$

Computation of the Cardinalities of Idempotents in Pf_n for $n = 2, 3, \dots, 10$

The above formula in proposition 5 is applied to compute the following table of values for the number of idempotents in Pf_n as shown in table 2

Table 2: Computations of the cardinality of idempotent elements in Pf_n based on rank using Stirling numbers.

n	r	$\binom{n}{r} r!$	$S(n-r, r)$	$\binom{n}{r} S(n-r, r)$	$\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} r! S(n-r, r)$
2	1	$\frac{2!}{1!} = 2$	$S(1,1) = 1$	2	2
3	1	$\frac{3!}{2!} = 3$	$S(2,1) = 1$	3	3
4	1	$\frac{4!}{3!} = 4$	$S(3,1) = 1$	4	
4	2	$\frac{4!}{2!} = 12$	$S(2,2) = 1$	12	16
5	1	$\frac{5!}{4!} = 5$	$S(4,1) = 1$	5	
5	2	$\frac{5!}{3!} = 20$	$S(3,2) = 3$	60	65
6	1	$\frac{6!}{5!} = 6$	$S(5,1) = 1$	6	
6	2	$\frac{6!}{4!} = 30$	$S(4,2) = 7$	210	
6	3	$\frac{6!}{3!} = 120$	$S(3,3) = 1$	120	336
7	1	$\frac{7!}{6!} = 7$	$S(6,1) = 1$	7	
7	2	$\frac{7!}{5!} = 42$	$S(5,2) = 15$	630	
7	3	$\frac{7!}{4!} = 210$	$S(4,3) = 6$	1260	1897
8	1	$\frac{8!}{7!} = 8$	$S(7,1) = 1$	8	
8	2	$\frac{8!}{6!} = 56$	$S(6,2) = 31$	1736	
8	3	$\frac{8!}{5!} = 336$	$S(5,3) = 23$	8400	
8	4	$\frac{8!}{4!} = 1680$	$S(4,4) = 1$	1680	11,824
9	1	$\frac{9!}{8!} = 9$	$S(8,1) = 1$	9	
9	2	$\frac{9!}{7!} = 72$	$S(7,2) = 63$	4536	

9	3	$\frac{9!}{6!} = 504$	$S(6,3) = 90$	45,360	
9	4	$\frac{9!}{5!} = 3024$	$S(5,4) = 10$	30,240	80,145
10	1	$\frac{10!}{9!} = 10$	$S(9,1) = 1$	10	
10	2	$\frac{10!}{8!} = 90$	$S(8,2) = 127$	11,430	
10	3	$\frac{10!}{7!} = 720$	$S(7,3) = 301$	216,720	
10	4	$\frac{10!}{6!} = 5040$	$S(6,4) = 65$	327,600	
10	5	$\frac{10!}{5!} = 30240$	$S(5,5) = 1$	30,240	586,000

D. Regularity of Pf_n

A semigroup S is said to be regular if for all $f \in S$ there exists an element $g \in S$ such that

$$fgf = f \quad (10)$$

Here the element g is not a group inverse. It is just an inverse in the semigroup sense.

In this section, we establish the concept of regularity in Pf_n . This is done by the construction of a Perfect-preserving pseudo-inverse which is different from the generalized group inverse. One distinguishing feature about the regularity of the transformations in Pf_n is that every transformation can have more than one pseudo-inverse. Thus unlike the generalized group inverse, the pseudo-inverse is not unique in the semigroup Pf_n .

Proposition 6

The set of perfect preserving full transformations is a regular semigroup.

Proof:

We shall prove by constructing a pseudo-inverse for every transformation in Pf_n which also satisfies the perfect preserving condition. Let $f \in Pf_n$ and let $Im(f) = \{y_1, y_2, \dots, y_r\}$ where $r \leq \lfloor n/2 \rfloor$. Define the kernel block of f as $A_i = f^{-1}(y_i)$ with $|A_i| \geq 2$. Choose a representative of the kernel block. i.e., pick one element $a_i \in A_i$ for each block. Thus $f(a_i) = y_i$. Next, define the pseudo-inverse g on the image y_i such that $g(y_i) = a_i$. This guarantees that $f(g(y_i)) = y_i$. However, this definition leaves some domain elements of g unmapped to any image, thereby rendering it a non-perfect-preserving transformation. To make g a perfect preserving transformation, the remaining unmapped domain elements of g are then assigned to the same set of image elements a_i of g in any permutation. i.e., we must define $g(y_i)$ for all i 's such that $|g^{-1}(a_i)| \geq 2$ for every image element $a_i \in g$ because $r \leq \lfloor n/2 \rfloor$. This is possible because there are always enough domain elements of g to ensure that every kernel block of g contains at least two elements. Thus we assign additional element so that $|g^{-1}(a_i)| \geq 2$ for each i . Hence $g \in Pf_n$.

Lastly we verify the regularity condition. Take any $x \in X_n$, suppose $f(x) = y_i$, then

$$g(f(x)) = g(y_i) = a_i \text{ and } f(a_i) = y_i. \text{ Thus } f(g(f(x))) = y_i = f(x).$$

Therefore, Pf_n is a regular semigroup. $\square$

Note: The assignment of the remaining elements to the image set of g can be done in multiple ways, depending on the permutation chosen. Hence unlike the generalized inverse, the pseudo-inverse is not unique.

Illustrative Example to Show Regularity of an Idempotent using Pf_4 .

Consider the idempotent $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} \in Pf_4$. $Im(f) = \{1,3\}$ and kernel blocks = $\{\{1,2\}, \{3,4\}\}$. This implies that; $f(1) = f(2) = 1$ and $f(3) = f(4) = 3$

Now in constructing the pseudo-inverse say g , let $g(1) = 2$ and $g(3) = 4$

This implies that $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & - & 4 & - \end{pmatrix}$. Now $g(1) = 1$ and $g(3) = 4$ but $g(2)$ and $g(4)$ are unmapped. Hence to make g a perfect preserving transformation, assign the remaining unmapped domain elements of g to the same image set of g in any permutation. Say,

$$g(2) = 2 \text{ and } g(4) = 4. \text{ Then, } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix} \in Pf_4$$

Verification;

$$f \circ g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} = f \quad (11)$$

Similarly, if we assign $g(2) = 4$ and $g(4) = 2$. Then we get another pseudo-inverse say h that satisfies the above condition as g ,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix} \in Pf_4$$

$$f \circ h \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} = f \quad (12)$$

This implies that f has more than one pseudo-inverse.

Consequently, every perfect-preserving transformation can have more than one pseudo-inverse.

Illustrative Example to Show Regularity of a Non-idempotent using Pf_4 .

Consider the non-idempotent element $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} \in Pf_4$.

$Im(f) = \{3,4\}$ and kernel blocks = $\{\{1,2\}, \{3,4\}\}$. This implies that; $f(1) = f(2) = 3$ and $f(3) = f(4) = 4$.

Now in constructing the pseudo inverse g , let $g(3) = 1$ and $g(4) = 3$

This implies that $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & 1 & 3 \end{pmatrix}$.

Now $g(3) = 1$ and $g(4) = 3$ but $g(1)$ and $g(2)$ are unmapped. Hence to make g a perfect preserving transformation, assign the remaining unmapped domain elements of g to the same image set of g in any permutation. Say, $g(1) = 1$ and $g(2) = 3$

$$\text{Then } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \in Pf_4$$

If assume $g(1) = 3$ and $g(2) = 1$ then this would yield another pseudo inverse say;

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix} \in Pf_4.$$

Verification:

$$f \circ g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} = f \quad (13)$$

$$f \circ h \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix} = f \quad (14)$$

Thus, even the non-idempotent elements have more than one pseudo-inverse.

Therefore, both the idempotent and non-idempotent elements in Pf_n are regular. Hence elements of Pf_n are regular.

E. Rank based Ideals in Pf_n

An ideal in a semigroup is a subset that absorbs multiplication from both left and right side. Ideals are the semigroup analogue of normal subgroups in group theory but not exactly, since semigroups may lack identity element. In full transformation semigroup, the most important ideals are defined by rank (size of image set). This is also the same in perfect-preserving transformation semigroup Pf_n . In Pf_n , the rank of a transformation say f is $|Im(f)|$. Let r be the rank of f , since each image point has at least 2 preimages; then $2r \leq n$, so the possible rank is $r = 1, 2, \dots, \lfloor n/2 \rfloor$. Thus we define an ideal of rank r in Pf_n as:

$$I_r = \{f \in Pf_n : \text{rank}(f) \leq r\} \quad (15)$$

Consequently, the minimal ideal I_1 is defined as

$$I_1 = \{f \in Pf_n : \text{rank}(f) = 1\} \quad (16)$$

which are just the constant transformations in Pf_n .

Proposition 7

Any ideal of rank r in Pf_n is a two-sided ideal. where $r = 1, 2, \dots, \lfloor n/2 \rfloor$.

Proof:

Let I_r be a left ideal ; $r = 1, 2, \dots, \lfloor n/2 \rfloor$. Take $f \in I_r$ such that $\text{rank}(f) \leq r$ and any $s \in Pf_n$. Then, $\text{rank}(s \circ f) \leq \text{rank}(f) \leq r$ because composition cannot increase rank. So, $s \circ f \in I_r$.

Similarly, let I_r be a right ideal ; $r = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. Take $f \in I_r$ such that $\text{rank}(f) \leq r$ and any $s \in Pf_n$. Then $\text{rank}(f \circ s) \leq \text{rank}(f) \leq r$. So, $f \circ s \in I_r$. Therefore, I_r is a two sided ideal. $\square$

Chain of Ideals in Pf_n .

From the proof of the proposition 7, it follows that Pf_n has a chain of ideals defined as follows;

$$I_1 \subset I_2 \subset I_3 \subset \dots \subset I_{\lfloor n/2 \rfloor} = Pf_n \quad (17)$$

where $\lfloor n/2 \rfloor = r$, the rank of a transformation in Pf_n and I_1 is the minimal ideal which consists of only constant maps with rank = 1.

For example;

In Pf_6 , $n = 6$. This implies $r = 1, 2, 3$. Thus the chain of ideal in Pf_6 are as follows.

$$I_1 \subset I_2 \subset I_3 = Pf_6 \quad (18)$$

$$I_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 5 & 5 & 5 & 5 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \right\}$$

$|I_1| = 6$ constant transformations.

$I_2 = \{f \in Pf_6 : \text{rank}(f) \leq 2\} = \text{all constant maps} + \text{all rank-2 maps}.$

$|I_2| = 6 \text{ constant maps} + 750 \text{ rank-2 maps} = 756 \text{ maps}.$

$I_3 = \{f \in Pf_6 : \text{rank}(f) \leq 3\} = \text{all constant maps} + \text{all rank-2 maps} + \text{all rank-3 maps}$

$|I_3| = 6 \text{ constant maps} + 750 \text{ rank-2 maps} + 1800 \text{ rank-3 maps} = 2556 \text{ maps} = |Pf_6|.$

These cardinalities of each rank based ideal is explicitly shown in table 2.

To show that $I_2 \subset Pf_6$ is an ideal, let us compose an element $f \in I_2$

with any element $s \in Pf_6$. Assume $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 4 & 4 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 1 & 3 & 4 \end{pmatrix}$

Left ideal;

$$s \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 4 & 4 & 1 & 1 & 1 \end{pmatrix} \in I_2$$

Right ideal;

$$f \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 3 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 3 & 4 & 3 & 3 & 4 \end{pmatrix} \in I_2$$

Since $s \circ f$ and $f \circ s \in I_2$, it implies that I_2 is indeed an ideal in Pf_6 .

F. Green's relations on Pf_n

Just like in T_n , the Green's relations are defined in Pf_n as follows. Let $f, g \in Pf_n$ (all functions satisfying the preimage condition from a set $X = \{1, 2, \dots, n\}$ to itself) where the operation of composition on transformation is performed from right to left, then;

$\mathcal{L}$ -relation: $f \mathcal{L} g$ if and only if $\ker(f) = \ker(g)$

$\mathcal{R}$ -relation: $f \mathcal{R} g$ if and only if $\text{Im}(f) = \text{Im}(g)$

$\mathcal{D}$ -relation: $f \mathcal{D} g$ if and only if $\text{rank}(f) = \text{rank}(g)$ just like in T_n .

$\mathcal{J}$ -relation: $\mathcal{J}$ -relation = $\mathcal{D}$ -relation

H-relation: $H = \mathcal{L} \cap \mathcal{R}$

The $\mathcal{D}$ -Classes

The $\mathcal{D}$ -classes just as in T_n are the possible ranks of elements in Pf_n .

Let $r = |\text{Im}(f)|$. Then $\sum_{y \in \text{Im}(f)} |f^{-1}(y)| \geq 2r$; $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Therefore, the $\mathcal{D}$ -classes of Pf_n is defined as :

$$D_r = \{ f \in Pf_n : |\text{Im}(f)| = r \} \quad \text{where } 1 \leq r \leq \lfloor \frac{n}{2} \rfloor.$$

$$|D_r| = \binom{n}{r} r! S_{n \geq 2}(n, r) \quad (19)$$

For example, the $\mathcal{D}$ -class in Pf_4 is obtained as shown below.

Let $X = \{1, 2, 3, 4\}$. $Pf_4 = \{ f : T_4 : |f^{-1}(y)| \geq 2 \text{ for every } y \in \text{Im}(f) \}$. Since $n = 4$, the possible rank are $r \leq \lfloor \frac{4}{2} \rfloor$. This implies that $r \leq 2$. If $r = 1$, then we have the D_1 class which consist of the constant maps having rank as 1. If $r = 2$, then we have the D_2 class which consist of the non-constant transformations of rank 2.

$$D_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

$$D_2 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 3 \end{pmatrix} \right\}$$

The $\mathcal{L}$ -Classes (Kernel classification)

The $\mathcal{L}$ -classes are transformations with same kernel partition. In T_n , $f \mathcal{L} g$ if and only if $\ker(f) = \ker(g)$. This also holds in Pf_n but the kernel partitions must satisfy the condition that each block has size ≥ 2 .

Thus, the kernel partitions X into r blocks each of size ≥ 2 . This exactly corresponds to Stirling numbers of the second kind with no singleton blocks.

Let us define a fixed partition of X as;

$\alpha = \{ \{A_1\}, \{A_2\}, \dots, \{A_r\} \}$ such that $|A_r| \geq 2$ where $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$ with $A_i \cap A_j = \emptyset$ for all $i, j \in r$ and $A_i \cup A_j = X$. Then the $\mathcal{L}$ -class is defined as follows:

$$L_\alpha = \{ f \in Pf_n : \ker(f) = \alpha \} \quad (20)$$

Thus, the number of $\mathcal{L}$ -classes in D_r which is equivalent to the number of partitions of X is given as

$$|L_\alpha| = S_{n \geq 2}(n, r) \quad (21)$$

where $S_{n \geq 2}(n, r)$ is the restricted Stirling numbers of the second kind with no singleton block.

For example, the $\mathcal{L}$ -class in Pf_4 are obtained as follows:

The $\mathcal{L}$ -classes in Pf_4 is simply the partition of $X = \{1, 2, 3, 4\}$. Here, the possible partitions as follows:

In D_1 , we have only one partition $\{1, 2, 3, 4\}$ for the four constant maps in Pf_4 .

The partition $\{1, 2, 3, 4\}$:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

In D_2 , there are three possible partition, $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$. Each partition represent one $\mathcal{L}$ -class. Thus there are three $\mathcal{L}$ -classes and each class consist of 12 transformations.

The partition $\{\{1, 2\}, \{3, 4\}\}$ has the following elements:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 3 \end{pmatrix} \right\}$$

The partition $\{\{1, 3\}, \{2, 4\}\}$ has the following elements:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 4 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{pmatrix} \right\}$$

The partition $\{\{1, 4\}, \{2, 3\}\}$ has the following elements:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 3 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \right\}$$

For example, consider the element

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \in Pf_4.$$

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix} \in \text{L-class } \{\{1, 2\}, \{3, 4\}\}.$

Perform left Multiplication.

$$s \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix} \in \text{L-class } \{\{1, 2\}, \{3, 4\}\}.$$

$$s \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix} \in \text{L-class } \{\{1, 2\}, \{3, 4\}\}.$$

Therefore, for any $s \in Pf_4$; $s \neq \text{constant map}$,

$$\text{Ker}(s \circ f) = \text{Ker}(s \circ g) \quad (22)$$

And $\text{Ker}(s \circ f), \text{Ker}(s \circ g) \in \text{L-class } \{\{1, 2\}, \{3, 4\}\}.$

The $\mathcal{R}$ -classes (Image classification)

The $\mathcal{R}$ -classes are transformations with same image. They might have different kernel but must satisfy the preimage condition $|f^{-1}(y)| \geq 2$ for all $y \in \text{Im}(f)$.

For a fixed $Y \subseteq X$, $|Y| = r$

Which is valid if we can partition n elements into r blocks of size ≥ 2 . So image choices still exist but have fewer transformations per image. Thus for a fixed subset $Y \subseteq X$, $|Y| = r$, then the $\mathcal{R}$ -class is defined as follows:

$$R_Y = \{ f \in Pf_n : \text{Im}(f) = Y \} \quad (23)$$

Consequently, the number of $\mathcal{R}$ -classes in D_r is obtain as follows:

$$|R_Y| = \binom{n}{r} \quad (24)$$

For example, the $\mathcal{R}$ -class in Pf_4 is described as follows:

The $\mathcal{R}$ -classes for the D_1 class are just the 4 constant transformations all having the image set $\{1,2,3,4\}$. Thus, there are 4 $\mathcal{R}$ -classes in the D_1 class of Pf_4 .

$$D_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

The $\mathcal{R}$ -classes for the D_2 class are all the transformations having image sets as follows $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$ and $\{3,4\}$. Each image set is equivalent to one $\mathcal{R}$ -class. Thus, there are a total 6 $\mathcal{R}$ -classes in the D_2 class of Pf_4 . There are 6 transformations in each image set as shown below.

For the image set $\{1,2\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix} \right\}$$

For the image set $\{1,3\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 1 & 3 \end{pmatrix} \right\}$$

For the image set $\{1,4\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 4 \end{pmatrix} \right\}$$

For the image set $\{2,3\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 3 \end{pmatrix} \right\}$$

For the image set $\{2,4\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 2 & 4 \end{pmatrix} \right\}$$

For the image set $\{3,4\}$ we have the following transformations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \right\}$$

For example, consider an element

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} \in Pf_4. \text{ Let } f, g \in \mathcal{R}\text{-class } \{1,2\} \text{ in the } D_2.$$

$$\text{Let } f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix} \in \mathcal{R}\text{-class } \{1,2\} \text{ in the } D_2.$$

Perform right Multiplication.

$$f \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 2 \end{pmatrix} \in \mathcal{R}\text{-class } \{1,2\} \text{ in the } D_2.$$

$$g \circ s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \end{pmatrix} \in \mathcal{R}\text{-class } \{1,2\} \text{ in the } D_2.$$

Therefore, for any $s \in Pf_4$; $s \neq \text{constant map}$,

$$\text{Im}(f \circ s) = \text{Im}(g \circ s) \quad (25)$$

And $\text{Im}(f \circ s), \text{Im}(g \circ s) \in \mathcal{R}\text{-class } \{1,2\}$ in the D_2 .

The $\mathcal{H}$ -classes (intersection of $\mathcal{L}$ and $\mathcal{R}$)

The $\mathcal{H}$ -classes are transformations which consist of the intersection between elements of the $\mathcal{L}$ -classes and the $\mathcal{R}$ -classes. i.e. the transformations having the same image and the same kernel with no singleton blocks.

Let image set $Y = \{y_1, y_2, \dots, y_r\}$ and kernel $\alpha = \{\{A_1\}, \{A_2\}, \dots, \{A_r\}\}$ where $|A_i| \geq 2$. Since f is constant on each block A_i and $\text{Im}(f) = r$, then each block must map to a distinct image element of Y . So, $f(A_i) = y_{\sigma(i)}$ for some permutation $\sigma \in S_r$. Each permutation defines a unique function.

$$f_\sigma(x) = y_{\sigma(i)} \quad \forall x \in A_i \quad (26)$$

Thus, the $\mathcal{H}$ -class parametrised by the image set Y and the kernel α with respect to a D_r class is defined as;

$$H_{(\alpha,Y)}(D_r) = \{f_\sigma \in Pf_n : \sigma \in S_r\} \quad (27)$$

Hence the number of transformations in each $\mathcal{H}$ -class of a D_r class is given as:

$$|S_r| = r! \quad (28)$$

Since $H = L_\alpha \cap R_Y$, then the number of H -classes in a D_r class is obtained as:

$$|H_{(\alpha,Y)}(D_r)| = S_{n \geq 2}(n, r) \binom{n}{r} \quad (29)$$

This justifies the earlier formula for the order of a D_r class.

$$|D_r| = \binom{n}{r} r! S_{n \geq 2}(n, r) \quad (30)$$

For example, the $\mathcal{H}$ -class in Pf_4 is described as follows:

Consider the D_1 class in Pf_4 , the single kernel block must map to one image element.

So, $A_1 \rightarrow a$

The number of $\mathcal{H}$ -class is:

$$|H_{(\alpha,Y)}(D_1)| = S_{n \geq 2}(4, 1) \binom{4}{1} = (1)(4) = 4 \quad (31)$$

For each image set, we obtain one $\mathcal{H}$ -class in each kernel block. There is only one kernel block and 4 image set in the D_1 class of Pf_4 . Hence there are 4 $\mathcal{H}$ -classes in the D_1 class of Pf_4 and each $\mathcal{H}$ -class has only one element. The one kernel block for all image set $\{1\}, \{2\}, \{3\}$ and $\{4\}$ is $\{1,2,3,4\}$. Since $r = 1$, then $|S_1| = 1! = 1$. Hence there is only 1 element in each $\mathcal{H}$ -class. Thus we have the following as our $\mathcal{H}$ -classes.

$$\begin{aligned} \{1,2,3,4\} \rightarrow 1 &= f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ \{1,2,3,4\} \rightarrow 2 &= f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix} \\ \{1,2,3,4\} \rightarrow 3 &= f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix} \\ \{1,2,3,4\} \rightarrow 4 &= f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} \end{aligned}$$

Hence the four H-classes are just the four constant maps of Pf_4 .

Now considering the D_2 class in Pf_4 , each block must map to a distinct image element.

So, $A_1 \rightarrow a, A_2 \rightarrow b$ or $A_1 \rightarrow b, A_2 \rightarrow a$

The number of H-class is

$$|H_{(\alpha,Y)}(D_2)| = S_{n \geq 2}(4,2) \binom{4}{2} = (3)(6) = 18 \quad (32)$$

For each image set, we obtain three $\mathcal{H}$ -classes in each kernel partition. There are three kernel partition and six image set in the D_2 class of Pf_4 . Hence there are 18 $\mathcal{H}$ -classes in the D_2 class of Pf_4 . Since $r = 2$, then $|S_2| = 2! = 2$. Hence there are 2 elements in each $\mathcal{H}$ -class.

In an arbitrary $\mathcal{H}$ -class, Let the image set be $Y = \{1,2\}$ and kernel Partition be $\alpha = \{\{1,2\}, \{3,4\}\}$ then ,

$\{1,2\} \rightarrow 1 \quad \{3,4\} \rightarrow 2$ or $\{1,2\} \rightarrow 2 \quad \{3,4\} \rightarrow 1$

Thus the two permutations in S_2 are;

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

Similarly if the image set is $Y = \{2,3\}$ and kernel Partition is $\alpha = \{\{1,3\}, \{2,4\}\}$ then,

$\{1,3\} \rightarrow 2 \quad \{2,4\} \rightarrow 3$ or $\{1,3\} \rightarrow 3 \quad \{2,4\} \rightarrow 2$

Thus the two permutations In S_2 are; $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 2 \end{pmatrix}$

In summary, for Pf_4 we have 2 D-classes; D_1 and D_2

In the D_1 class, we have the following:

1 $\mathcal{L}$ -class,

4 $\mathcal{R}$ -classes, and

4 $\mathcal{H}$ -classes

In the D_2 class, we have the following:

3 $\mathcal{L}$ -classes,

6 $\mathcal{R}$ -classes, and

18 $\mathcal{H}$ -classes

G. Relationship between Perfect-preserving transformation Pf_n and Singular transformation $Sing_n$.

A transformation $f: \{1,2,\dots,n\} \rightarrow \{1,2,\dots,n\}$ is called singular if it is not bijective. That is, f is not a permutation. The symmetric group S_n is the set of transformations that are permutation. Therefore, singular transformation is the full transformation T_n without the symmetric group S_n . Thus,

$$Sing_n = T_n \setminus S_n \quad (33)$$

$$Sing_n \cap S_n = \emptyset \quad (34)$$

$$|Sing_n| = n^n - n! \quad (35)$$

From the observation explained in section about the categorical exclusion of injective mappings in perfect-preserving transformations, we conclude without fear that Pf_n consist of transformations that are not bijective. Thus,

$$Pf_n \cap S_n = \emptyset \quad (36)$$

Therefore the element of Pf_n are all singular transformations. However, note that not all singular transformations are perfect-preserving. Thus, Pf_n is a proper subclass of $Sing_n$. i.e.

$Pf_n \subset Sing_n$. This implies that;

$$Pf_n \cap Sing_n = Pf_n \quad (37)$$

Proposition 8

For any integer $n \geq 2$, if $f: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ is perfect preserving, then f is singular. i.e. Every perfect-preserving transformation is singular.

Proof:

Recall the Perfect preserving preimage condition of Pf_n and the non-bijectivity of singular transformation. For a function from a finite set to itself, non-injective is equivalent to not injective or not surjective. Thus, it suffices to prove that f is not injective. Assume for contradiction that f is injective, then for every $y \in Im(f)$, the preimage $f^{-1}(y)$ contains exactly one element. Since distinct domain elements map to distinct image elements. Thus,

$$|f^{-1}(y)| = 1 \text{ for all } y \in Im(f) \quad (38)$$

But the perfect-preserving condition requires that $|f^{-1}(y)| \geq 2$ for all $y \in Im(f)$ and the injectivity assumption gives $|f^{-1}(y)| = 1$. For these two condition to both hold, we would need $1 \geq 2$ which is impossible. Therefore f cannot be injective. Thus f is not bijective. Hence f is singular. $\square$

IV. CONCLUSION

This paper investigated the algebraic structure of the perfect-preserving full transformation semigroup Pf_n , a subsemigroup of the full transformation semigroup T_n defined by the requirement that every element of the image has at least two preimages. The imposed fibre condition was shown to have significant structural consequences. In particular, every transformation in Pf_n has rank at most $\lfloor n/2 \rfloor$, which excludes all injective transformations and establishes that Pf_n consists entirely of singular transformations. It was also established that non-constant perfect-preserving transformations exist if and only if $n \geq 4$. The study further demonstrated that Pf_n is closed under composition and hence forms a semigroup. Its idempotents were characterized by the property that every fixed point has at least one additional preimage. Regularity was established through the construction of a perfect-preserving pseudo-inverse for each element, showing that the inverse required for regularity can be constructed within the restricted semigroup. The analysis of rank also led to a chain of two-sided ideals, with the constant transformations constituting the minimal ideal. Finally, the Green-class structure of Pf_n was described using the kernel and image of transformations. The admissible kernel partitions consist exclusively of blocks of size at least two, while the corresponding image classes are determined by admissible image sets. For a fixed kernel and image, the associated $\mathcal{H}$ -class is determined by the possible assignments of kernel blocks to image points. Upon comparison with singular transformation, results from this paper established that every element of Pf_n is singular and has rank at most $\lfloor n/2 \rfloor$. These results demonstrate that the perfect-preserving condition produces a distinctive algebraic structure within T_n and provide a foundation for further investigations into the enumeration, conjugacy classes, order-preserving subsemigroups, and other combinatorial properties of Pf_n .

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