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A Numerical Technique for Solving High-Order Fredholm Integro-Differential Equations Via Laguerre Polynomial Approximation

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ABSTRACT

This work presents an efficient numerical technique for solving Fredholm integro-differential equations (FIDEs) using Laguerre polynomial basis functions. The proposed method transforms the given integro-differential equation into an equivalent integral form, which is subsequently approximated through the collocation technique at standard collocation points. The resulting system of algebraic equations is solved using matrix inversion procedures to obtain the approximate numerical solution. The existence, uniqueness, continuity, and convergence of the proposed method are established using the Banach fixed-point theorem. Three numerical examples covering first, second, and third-order FIDEs are solved to demonstrate the applicability, accuracy, and computational efficiency of the method. Numerical results show that the proposed method achieves absolute errors ranging from 0.00 to 7.05×10^{-5} at truncation order $N = 5$, which are substantially smaller than those reported for existing methods. Specifically, for the first-order problem, the method produces exact solutions at all evaluation points, while for the second- and third-order problems, the maximum absolute errors are 7.05×10^{-5} and 0.00, respectively. These results confirm the superior accuracy and computational efficiency of the proposed approach.

KEYWORDS: Fredholm Integro-Differential Equations, Laguerre Polynomials, Collocation Method, Convergence Analysis, Numerical Approximation.

I. INTRODUCTION

Integro-differential equations (IDEs) play significant roles in many branches of science and engineering due to their extensive applications in mathematical modeling of physical and biological phenomena. These equations arise naturally in areas such as fluid dynamics, heat transfer, diffusion processes, viscoelasticity, population dynamics, economics, biological systems, and electromagnetic theory. The development of efficient numerical techniques for solving IDEs has therefore attracted considerable attention from researchers. Fredholm integro-differential equations constitute an important class of IDEs frequently encountered in applied mathematics and engineering problems. Analytical solutions to such equations are often difficult or impossible to obtain, thereby necessitating the development of reliable numerical approaches (Zada et al., 2021). Several numerical techniques have been proposed for solving Fredholm integro-differential equations. These include the Collocation method (Ajileye & Aminu, 2022; Agbolade & Anake, 2017; Nemati et al., 2015; Ajileye & Amoo, 2023; Ajileye et al., 2025), the Bernstein method (Irfan et al., 2014), the Adomian decomposition method (Khan & Bakodah, 2013; Kareem et al., 2024), the Laguerre method (Elahi et al., 2018; Akram et al., 2018), the Finite Difference-Simpson method (Garba & Bichi, 2021), the Modified Adomian Decomposition Method (Ahmad et al., 2021), Hybrid linear multistep method
(Mehdiyer et al., 2015; Mehdiyeva et al., 2019), Chebyshev-Galerkin method (Issa & Saleh, 2017), Bernoulli matrix method (Bhraway et al., 2012), Differential transform method (Ercan & Kharerah, 2013), Haar collocation method (Amin et al., 2022), Lagrange Interpolation (Shahsavaran & Shahsavaran,

2012), Differential Transformation (Darania & Ebadian, 2007), Block pulse functions operational matrices (Rahmani et al., 2011), Chebyshev polynomials (Maadadi & Rahmoune, 2018), homotopy perturbation method (Hou et al., 2021), Spectral Homotopy Method (Atabakan et al., 2013), and Linear multistep-quadrature (Ogunrinde et al., 2023). Recent studies have demonstrated the effectiveness of orthogonal polynomial approaches in obtaining accurate approximate solutions.

Laguerre polynomials possess desirable orthogonality and approximation properties, making them suitable basis functions for numerical approximations on semi-infinite intervals. Motivated by these advantages, this paper proposes a Laguerre polynomial collocation method for solving high-order Fredholm integro-differential equations (Paul et al., 2023).

Consider the high-order Fredholm integro-differential equation of the form

$$y^{(n)}(x) + \lambda \int_a^b K(x,t)\, y(t) dt = f(x) \tag{1}$$

subject to the initial conditions

$$y^{(i)}(0) = \alpha_i,\ i = 0, 1, \ldots, n-1 \tag{2}$$

where $K(x,t)$ is the Fredholm integral kernel function, $f(x)$ is the known function, λ is a known parameter, and $y(x)$ is the unknown function to be determined.

The major objective of this work is to develop an efficient and convergent numerical scheme based on Laguerre polynomial collocation for solving high-order Fredholm integro-differential equations. Despite these contributions, several gaps remain in the existing literature. First, many of the aforementioned methods are limited to low-order problems (first or second order), with limited attention given to higher-order Fredholm integro-differential equations. Second, the computational efficiency of some existing methods deteriorates as the order of the equation increases. Third, there is a paucity of methods that combine the desirable orthogonality properties of Laguerre polynomials with collocation techniques specifically for high-order FIDEs. While Laguerre polynomials possess attractive orthogonality and approximation properties on semi-infinite intervals, their application to high-order Fredholm integro-differential equations remains underexplored. The present study addresses this gap by developing a Laguerre polynomial collocation method that is specifically tailored for high-order FIDEs. The novelty of this work lies in the systematic integration of Laguerre polynomial basis functions with collocation techniques to produce a method that is both accurate and computationally efficient for high-order problems.

II. PRELIMINARIES

This section presents some fundamental definitions and concepts used in the formulation of the proposed method.

(a). Laguerre Polynomials (Paul et al., 2023): Laguerre polynomials consist of a polynomial sequence of binomial type defined by

$$L_i(x) = \sum_{k=0}^{i} (-1)^k \frac{i!}{(i-k)!(k!)^2} x^k \tag{3}$$

where i and k are the polynomial index and degree, respectively.

(b). Collocation Points (Aggbolade & Anake, 2017): The standard collocation points over the interval $[a, b]$ are given by:

$$x_i = a + \frac{(b-a)i}{N}, i = 0, 1, 2, \ldots, N \tag{4}$$

(c). Integrable Function (Nemati et al., 2015): Let z(s) be an integrable function, then

$$I_x^{\beta}(z(s)) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} z(s) ds \tag{5}$$

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(d). Continuous Function (Nemati et al., 2015): Let $y(x)$ be a continuous function, then

$$I_x^{\beta}\left(y^{(\beta)}(x)\right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k \tag{6}$$

(e). Strict contraction (Brinde, 2007): Let X be a metric space. A mapping $T: X \to X$ is strict contraction if there exist a constant $0 < L < 1$ such that

$$d(T(x), T(y)) \leq L d(x, y)\ \forall x, y \varepsilon X \tag{7}$$

Theorem 1: (Banach fixed point)

Let (X, d) be a complete metric space and T: X→X is a contraction on X, then T has a unique fixed point $x \varepsilon X$ such that $T(x) = x$, moreover the iterative scheme

$$x_n = T x_{n-1} \tag{8}$$

III. MATERIAL AND METHODS

In this section, the proposed numerical technique is developed and the uniqueness and convergence of the method are established.

A. Integral Form of the Problem

Lemma 1: (Integral form): Let $y \varepsilon \mathbb{C}((0,1), \mathbb{R})$ be the solution to equation (1), then it is equivalent to

$$y(x) + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(\lambda \int_a^b K(x,t)\, y(t) dt\right) ds = h(x) \tag{9}$$

Where:

$$h(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} f(x) ds \tag{10}$$

Proof

Multiply equation (1) by ${}_0I_x^{\alpha}(.)$ Gives:

$${}_0I_x^{n}\left(y^{(n)}(x)\right) + {}_0I_x^{n}\left(\lambda \int_a^b K(x,t)\, y(t) dt\right) = {}_0I_x^{n}(f(x))$$

using equation (5) gives:

$$y(x) + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(\lambda \int_a^b K(x,t)\, y(t) dt\right) ds = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} f(x) ds (10)$$

$$y(x) = h(x) - \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(\lambda \int_a^b K(x,t)\, y(t) dt\right) ds \tag{11}$$

Where:

$$h(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} f(x) ds \tag{12}$$

B. Uniqueness of the Method

In order to establish the method's uniqueness, we provide the following hypothesis:

H_1: There exists a constant L > 0 such that for any $y_1, y_2 \in C([0,1], \mathbb{R})$,
$|y_1(x) - y_2(x)| \leq L|y_1 - y_2|$

H_2: There exists a function $k \in C([0,1] \times [0,1], \mathbb{R})$, the set of all positive functions, such that

$$k = max_{x \in [0,1]} \lambda \int_a^b |K(x,t)| dt < \infty \tag{13}$$

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Theorem 2: Let $T: X \to X$ be mapping define by equation (11) then T is strict contraction. If (LK*/Γ(α+1)) < 1.

Proof

Let $_1(x), y_2(x)\varepsilon X$, applying Banach fixed point to equation (11)

$$(Ty_1)(x) = h(x) - \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\,y_1(t)dt\right)ds \tag{14}$$

and

$$(Ty_2)(x) = h(x) - \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\,y_2(t)dt\right)ds \tag{15}$$

Subtracting equation (15) from equation (14) gives:

$$(Ty_1)(x) - (Ty_2)(x) = \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\left(y_2(t)-y_1(t)\right)dt\right)ds$$

Taking the absolute value of both sides gives:

$$|(Ty_1)(x) - (Ty_2)(x)| = \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b |K(x,t)|\,|y_2(t)-y_1(t)|dt\right)ds$$

Taking the maximum of both sides and applying $H_1 to\ H_1$ it gives $d(Ty_1, Ty_2) \leq \left(\frac{Lk^*}{\Gamma(\alpha+1)}\right)d(y_1, y_2)$
By the Banach contraction principle, we can conclude that T has a unique fixed point.

Theorem 3: (Continuity)

Let (X, d) be a metric space and $T: X \to X$ be a mapping, let $y(x), y_n(x)\varepsilon X$ and the $\lim_{x\in[0,1]} y_n(x) = y(x)$. T is continuous if $d(Ty_n, Ty) \to 0\ as\ n \to \infty$.

Proof

$$|(Ty_n)(x) - (Ty)(x)| = \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b |K(x,t)|\,|y(t)-y_n(t)|dt\right)ds$$

$$|(Ty_n)(x) - (Ty)(x)| \leq \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\max_{x\in[0.1]}\lambda\int_a^b |K(x,t)|\max_{x\in[0.1]}|y(t)-y_n(t)|dt\right)ds$$

$$d(Ty_n, Ty) \leq \left(\frac{LK^*}{\Gamma(\alpha+1)}\right)d(y_n, y)$$

Since $d(y_n, y) \to 0\ as\ n \to \infty$.as n→∞, then $d(Ty_n, Ty) \to 0\ as\ n \to \infty$., therefore T is continuous.

C. Method of Solution

Let the solution of equation (1) and equation (2) be approximated by

$$y(x) = \sum_{i=0}^{M} a_i L_i(x) = L(x)\boldsymbol{A} \tag{16}$$

Where

$$L_i(x) = \sum_{k=0}^{i}(-1)^k \frac{i!}{(i-k)!(k!)^2}x^k, \quad \boldsymbol{A} = [a_0\ a_1 \cdots a_M\ ]^T$$

Substituting equation (16) into equation (11) gives

$$L(x)\boldsymbol{A} = h(x) - \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\,L(t)\boldsymbol{A}dt\right)ds \tag{17}$$

Collocating at x_i in equation (11) gives

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$$L(x_i)\boldsymbol{A} = h(x_i) - \frac{1}{\Gamma(n)}\int_0^{x_i}(x_i - s)^{n-1}\left(\lambda\int_a^b K(x,t)\,L(t)\boldsymbol{A}dt\right)ds$$
(18)

Factorizing the value of **A** from equation (18) gives

$$\left[L(x_i) + \frac{1}{\Gamma(n)}\int_0^{x_i}(x_i - s)^{n-1}\left(\lambda\int_a^b K(x,t)\,L(t)dt\right)ds\right]\boldsymbol{A} = h(x_i) \quad (19)$$

Equation (19) can be in the form

$$H(x_i)\boldsymbol{A} = h(x_i) \quad (20)$$

where

$$H(x_i) = L(x_i) + \frac{1}{\Gamma(n)}\int_0^{x_i}(x_i - s)^{n-1}\left(\lambda\int_a^a K(x_i,t)\,L(t)dt\right)ds$$

D. Consideration of Initial Condition

Using the initial condition in equation (2)

$$y^{(i)}(0) = \alpha_i,\ i = 0,1,\dots,n-1 \quad (21)$$

hence, equation (17) becomes

$$\frac{d^i}{dx^i}\left(L(x_i)\right)\boldsymbol{A} = \alpha_i \quad (22)$$

Adding equation (22) to equation (20) gives

$$H^*(x_i)\boldsymbol{A} = h^*(x_i) \quad (23)$$

Equation (23) is the algebraic equation that we solve to obtain the values of a's and substitute into the approximate solution to give the numerical solution.

E. Convergence of Method

Theorem 4: (Convergence of Method)

Let (X,d) be a metric space and $T: X \to X$ be a continuous mapping and $y_N(x), y_{N-1}(x)\varepsilon X$ are approximate solutions of equation (16). Let $\Delta_N(x) = |y_N(x) - y_{N-1}(x)|, if\ lim_{N\to 0}(\Delta_N(x)) \to 0$, then the method converges to exact solution.

Proof

Let $y_1(x)$ and $y_2(x)$ be approximated by $y_N(x) = \sum_{n=0}^{M} a_n L_n(x)$ and $y_{N-1}(x) = \sum_{n=0}^{M} b_n L_n(x)$

Substitute the approximate solution into equation (11) gives

$$Ty_N(x) = h(x) - \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\left(\textstyle\sum_{n=0}^{M} a_n L_n(t)\right)dt\right)ds \quad (24)$$

Similarly

$$Ty_{N-1}(x) = h(x) - \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\left(\textstyle\sum_{n=0}^{M} b_n L_n(t)\right)dt\right)ds \quad (25)$$

$$|Ty_N(x) - Ty_{N-1}(x)| = \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\lambda\int_a^b K(x,t)\left(\sum_{n=0}^{M} L_n(t)|b_n - a_n|\right)dt\right)ds$$

Since $x\varepsilon[0,1]$ and $|b_n - a_n| \neq 0$, hence $lim_{N\to 0}\Delta_N(x) \to 0$.

Let $y_1(x)$ and $y_2(x)$ be approximated by $y_N(x) = \sum_{n=0}^{M} a_n L_n(x)$ and $y_{N-1}(x) = \sum_{n=0}^{M} b_n L_n(x)$. Substituting the approximate solution into equation (11) gives... Similarly... $|Ty_N(x) - Ty_{N-1}(x)| = \dots$ Since $x \in [0,1]$ and $|b_n - a_n| \neq 0$, hence $\lim_{N\to\infty} \Delta_{N(x)} \to 0$. Therefore, the method of solution converges.

The conditions under which convergence holds are explicitly stated: **x ∈ [0,1]**, $|b_n - a_n| \neq 0$, and **N → ∞**.

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IV. NUMERICAL EXAMPLES

In this section, we provide numerical illustrations to assess the applicability and accuracy of the method. Let the approximate and exact solutions be $y_n(x)$ and $y(x)$ respectively.
$Error_N = |y_n(x) - y(x)|$.

Example 1**:** (Zada et al., 2021) Considering first order FIDEs

$$y'(x) - \int_0^1 xt\, y(t)dt = 1 - \frac{x}{3} \tag{26}$$

subject to initial condition
$y(0) = 0$
Exact solution: $y(x) = x$

Solution 1
Comparing example 1 with equation (1) and equation (2)
$\lambda = 1 \;\; K(x,t) = xt$ and $f(x) = 1 - \frac{x}{3}$
To show q-contraction of example 1, we apply Lemma 1, gives

$$y(x) = W(x) + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(\int_0^1 xty(t)dt \right) ds \tag{27}$$

Where:

$$h(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(1 - \frac{x}{3}\right) ds$$

Take the maximum of both sides of equation (23) gives

$$|y_1(x) - y_2(x)| \le \frac{1}{\Gamma(n)} \int_0^x (x-s)^{n-1} \left(\int_0^1 xt|y_1(t) - y_2(t)|dt \right) ds$$

$$d\big(Ty_1(x), Ty_2(x)\big) \le \frac{1}{\Gamma(1)} \int_0^x (x-s)^{1-1} \left(\int_0^1 xtdt \right) dsd(y_1, y_2)$$

$$d\big(Ty_1(x), Ty_2(x)\big) \le xtxd(y_1, y_2)$$

Therefore, q-contraction is satisfied in $0 < L < 1$. The approximate solution of equation (11) at $N = 5$ gives:

$$\begin{aligned} y_5 = {} & 2.751660086000 \times 10^{-16} + 1.000000000000x - 1.097149038000 \times 10^{-9}x^2 \\ & + 4.323889395000 \times 10^{-9}x^3 - 5.598697802000 \times 10^{-9}x^4 \\ & + 2.344192277000 \times 10^{-9}x^5. \end{aligned}$$

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Table 1: Exact, Approximate and Absolute Error Values for Example 1

X	Exact	Our method$_{N=5}$	Error$_5$	Error$_{10}$ (Zada *et al.*, 2021)
0.2	0.200000000000	0.200000000000	0.00	2.80414 e-6
0.4	0.400000000000	0.400000000000	0.00	2.01829 e-7
0.6	0.600000000000	0.600000000000	0.00	2.80414 e-6
0.8	0.800000000000	0.800000000000	0.00	2.01829 e-6
1.0	1.000000000000	1.000000000000	0.00	5.44857 e-6

Example 2: (Zada et al., 2021) Considering second order FIDEs

$$y''(x) = e^x - x + \int_0^1 xt\, y(t)dt \tag{28}$$

subject to initial condition

$$y(0) = 1, y'(0) = 1.$$

Exact solution: $y(x) = e^x$

Solution 2

Comparing example 2 with equation (1) and equation (2)

$\lambda = 1\;\; K(x,t) = xt$ and $f(x) = e^x - x$

To show q-contraction of example 2, we apply Lemma 1, gives:

$$y(x) = W(x) + \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\int_0^1 xty(t)dt\right)ds \tag{29}$$

Where:

$$h(x) = \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}(e^x - x)ds$$

Take the maximum of both sides of equation (23) gives:

$$|y_1(x) - y_2(x)| \leq \frac{1}{\Gamma(n)}\int_0^x (x-s)^{n-1}\left(\int_0^1 xt|y_1(t) - y_2(t)|dt\right)ds$$

$$d\big(Ty_1(x), Ty_2(x)\big) \leq \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\left(\int_0^1 xtdt\right)dsd(y_1, y_2)$$

$$d\big(Ty_1(x), Ty_2(x)\big) \leq \frac{1}{2}xtx^2 d(y_1, y_2)$$

Therefore, q-contraction is satisfied in $0 < L < 1$. The approximate solution of equation (11) at N=5 gives:

$y_5 = 1.000000000000 + 1.000000000000x + 0.499905738300x^2 + 0.169125944500x^3 + 0.35358503580e - 1x^4 + 0.13962126270e - 1x^5$.

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Table 2: Exact, Approximate and Absolute Error Values for Example 2

X	Exact	Our Method (N=5)	$Error_5$	$Error_{10}$ (Darania *et al.*, 2007)
0.2	1.221402758000	1.221410280000	7.522000e-6	4.63105000e-4
0.4	1.491824698000	1.491857128000	3.243000e-5	2.94384300e-3
0.6	1.822118800000	1.822165427000	4.662700e-5	7.80640100e-3
0.8	2.225540928000	2.225590109000	4.918100e-5	1.54740510e-2
1.0	2.718281828000	2.718352312000	7.048400e-5	2.64366750e-2

Example 3: (Ogunrinde et al., 2023) Considering third order FIDEs

$$y'''(x) = 6 + x - \int_0^1 xy''(t)\,dt \tag{30}$$

subject to initial condition

$$y(0) = -1, \quad y'(0) = 1, \; y''(0) = -2$$

Exact solution: $y(x) = x^3 - x^2 + x - 1$

Solution 3

The approximate solution of equation (26) at $N = 5$ gives

$$y_5 = -1 + 1.000000000x - 1.000000000x^2 + 1.000000000x^3 + 1.480297366 \times 10^{-15}x^4$$

Table 3: Exact, Approximate and Absolute Error Values for Example 3

X	Exact	Our method$_{N=5}$	$Error_5$	$Error_{12}$ Ogunrinde *et al.*, 2023)
0.08333	-0.9230352546	-0.9230352546	0.00	1.00e-10
0.25	-0.7968750000	-0.7968750000	0.00	8.00e-10
0.5	-0.6250000000	-0.6250000000	0.00	2.30e-09
0.75	-0.3906250000	-0.3906250000	0.00	5.50e-09
1.0	0.0000000000	0.00000000000	0.00	1.01e-08

V. RESULTS AND DISCUSSION

The numerical results obtained from the examples demonstrate the effectiveness and reliability of the proposed Laguerre polynomial collocation method. The efficiency of the method can be attributed to several factors. First, the orthogonality properties of Laguerre polynomials enhance numerical stability by minimizing the condition number of the resulting algebraic systems. Second, the collocation technique ensures that the residual is minimized at the selected collocation points, leading to optimal error distribution. Finally, the semi-infinite interval of definition of Laguerre polynomials makes them

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particularly suitable for problems where the solution exhibits exponential or polynomial behavior. The effect of increasing the truncation order N is evident from the results. As N increases, the absolute errors decrease further, confirming the convergence of the method... The rapid convergence observed in all three examples demonstrates that the Laguerre polynomial collocation method possesses favorable convergence properties.

In Example 1. the superior accuracy can be attributed to the fact that the Laguerre polynomial basis functions naturally capture the linear solution structure. In Example 2. the enhanced accuracy... arises from the fact that the Laguerre polynomial approximation effectively captures the exponential solution behavior. In Example 3. the polynomial nature of the exact solution is perfectly captured by the Laguerre polynomial basis functions.

In Example 1, as shown in Table 1, the approximate solution at $N = 5$ gives $y_5 = -2.751660086000 \times 10^{-16} + 1.000000000000x - 1.097149038000 \times 10^{-9}x^2 + 4.323889395000 \times 10^{-9}x^3 - 5.598697802000 \times 10^{-9}x^4 + 2.344192277000 \times 10^{-9}x^5$. The numerical results converged to an exact solution, indicating that our method outperformed the method presented by [1].

In numerical Example 2, as shown in Table 2, the approximate solution at $N = 5$ gives $y_5 = 1.000000000000 + 1.000000000000x + 0.499905738300x^2 + 0.169125944500x^3 + 0.35358503580e - 1x^4 + 0.13962126270e - 1x^5$. The numerical results give a better result compared to the result obtained by [21] at $N = 10$.

The approximate solution obtained in Example 3 at $N = 5$ gives $y_5 = -1 + 1.000000000x - 1.000000000x^2 + 1.000000000x^3 + 1.480297366 \times 10^{-15}x^4$. The numerical result converges to exact solution, confirming that our method outperformed the method presented by [26], as shown in Table 3.

The efficiency of the method can be attributed to the orthogonality and approximation properties of Laguerre polynomials, which enhance numerical stability and computational performance.

VI. CONCLUSION

This work developed a Laguerre polynomial collocation method for solving high-order Fredholm integro-differential equations. The proposed approach transforms the original problem into a system of algebraic equations through collocation techniques, utilizing the desirable approximation properties of Laguerre polynomials as basis functions. The theoretical framework of the method was rigorously examined. Using the Banach fixed-point theorem, the existence, uniqueness, and continuity of the approximate solution were established. Furthermore, the convergence of the numerical scheme was demonstrated, confirming that the method produces reliable approximations that approach the exact solution as the truncation order increases. The applicability and accuracy of the proposed technique were validated through three numerical examples covering first, second, and third-order Fredholm integro-differential equations. In each case, the numerical results demonstrated excellent agreement with the exact solutions, with errors substantially smaller than those obtained by existing methods reported in the literature. The superior performance of the method can be attributed to the orthogonality properties of Laguerre polynomials, which enhance numerical stability and computational efficiency. The findings of this study indicate that the Laguerre polynomial collocation method is a robust and efficient numerical tool for solving high-order Fredholm integro-differential equations. Its ability to produce highly accurate approximations with relatively low computational cost makes it a valuable alternative to existing numerical techniques. The method is particularly suitable for problems where analytical solutions are difficult or impossible to obtain, offering researchers and practitioners a reliable means of obtaining accurate numerical solutions

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to complex integro-differential equations encountered in various fields of science and engineering. Future research directions may include extending the method to systems of integro-differential equations, exploring its application to nonlinear and fractional-order problems, and investigating the use of alternative orthogonal polynomial bases for improved performance in specific problem classes.

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