

# A Stable and Efficient Second Derivative Block Hybrid Method for the Solution of Stiff Systems of Ordinary Differential Equations

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**ABSTRACT:** Stiff systems of Ordinary Differential Equations (ODEs) often appear in scientific and engineering applications where solutions exhibit rapid variation, making their study highly significant. Traditional numerical approaches, including explicit Runge–Kutta methods and several multistep schemes, usually face stability challenges when applied to such problems. To maintain stability, very small step sizes are required, which greatly increases computational cost. To address this limitation, this work proposes a new second-derivative block hybrid method designed for the efficient solution of stiff ODEs. The method relies on a power series expansion that is both interpolated and collocated at selected points to generate a continuous block scheme. By incorporating off-grid points together with second derivatives, the technique achieves high-order accuracy while ensuring zero-stability, consistency, and convergence. The developed scheme is characterized by a broad region of absolute stability and maintains a uniform order of twelve. Numerical experiments carried out on well-known stiff test problems highlight the reliability and efficiency of the method, producing errors far smaller than those obtained with conventional techniques. The results provide clear evidence of the method's ability to handle the computational difficulties associated with stiff systems and establish it as a strong alternative to existing approaches. This contribution enhances the progress of more advanced numerical strategies capable of performing effectively in real-world fields such as chemical reaction dynamics, systems biology, and control processes.

**KEYWORDS:** Stiff Systems of ODEs, Block Hybrid Method, Second Derivative, Numerical Stability, Power Series, Collocation Technique.

## I. INTRODUCTION

Stiff systems of ordinary differential equations (ODEs) frequently arise in complex scientific and engineering applications, making their understanding essential. These systems are particularly challenging for classical numerical methods, as they often exhibit rapid changes in certain components (Abhulimen & Igbokwe, 2017). Both one-step and multistep approaches generally require very small step sizes to maintain stability in stiff regions, which results in slower performance and higher computational costs (Yakubu et al., 2017). The main difficulty in solving stiff ODEs lies in the limited stability and efficiency of standard techniques, especially when the solution varies sharply over short intervals (Kuboye & Omar, 2015). Block hybrid methods, which compute several solution points simultaneously, have shown promise in improving both stability and efficiency. In addition, the use of second derivatives enhances accuracy and ensures faster convergence (Althemai et al., 2014). However, many existing methods have yet to fully exploit the potential of second-derivative formulations. This study seeks to address that gap by developing a second-derivative block hybrid method specifically designed for stiff ODEs. The aim is to deliver higher-order accuracy, improved stability, and computational efficiency while reducing the costs typically associated with implicit methods (Jator, 2014; Kamoh et al., 2021). While implicit techniques generally provide greater stability, they often do so at the expense of increased computational demand, which this work attempts to minimize.

This study focuses on the stiff systems of ordinary differential equations of the form:

$$\left. \begin{aligned} y'_1(x) &= f(x_1, y_1(x)), & y(x_1) &= y_1 \\ y'_2(x) &= f(x_2, y_2(x)), & y(x_2) &= y_2 \end{aligned} \right\} \quad (1.1)$$

The numerical treatment of stiff differential equations plays a vital role in disciplines such as chemical kinetics, systems biology, and control theory. Over the years, the challenge of solving stiff ordinary differential equations (ODEs) has encouraged the development of methods that emphasize both accuracy and efficiency, with particular attention to implicit and hybrid schemes. The main difficulty lies in the fact that stiff ODEs often display rapid variations, which render standard explicit approaches unstable unless very small step sizes are employed. Earlier hybrid methods, such as those presented by Awoyemi and Kayode (2016), relied on first derivatives but encountered limitations in achieving stability when applied to stiff problems. Progress was later made by incorporating second derivatives, which significantly improved both stability and accuracy. Notable contributions in this regard include Jator (2014) and Onumanyi et al. (2014), who introduced second-derivative multistep collocation techniques. These developments paved the way for more advanced approaches.

Subsequent studies by Kamoh (2021) and Abdul Azeez (2022) produced high-order implicit second-derivative block hybrid methods, demonstrating A-stability and superior performance on stiff test equations. More recently, Panopoulos and colleagues (2023) have focused on enhancing computational efficiency by reducing the cost of matrix inversions and introducing adaptive mechanisms for step size and order selection. Collectively, these efforts highlight the ongoing quest to balance the sophistication of numerical methods with their practical computational demands. Skwame et al. (2017) introduced hybrid block backward differentiation formulas of the second-derivative type with orders six and seven. Their approach improved the accuracy of stiff initial value problems by combining first and second derivatives within a multistep collocation framework. Building on this direction, Bakari and colleagues (2020) proposed a two-step hybrid block backward differentiation scheme aimed at solving stiff ordinary differential equations. Their method employed collocation and interpolation strategies that incorporated second derivatives to enhance both accuracy and stability. Further innovation came from Najamuddeen et al. (2022), who designed a three-point diagonally implicit block backward differentiation formula with a lower triangular coefficient matrix. This structure allowed the method to achieve greater stability while computing three solution points per step, resulting in fifth-order accuracy for stiff initial value problems. In a related development, Sabo and co-researchers (2018) presented a modified second-derivative linear multistep technique tailored to stiff ODEs, particularly those arising in mechanical systems and control theory. Numerical experiments showed that their approach outperformed many existing methods in terms of accuracy.

## II. DERIVATION OF SECOND DERIVATIVE BLOCK HYBRID METHODS

In this section, we will derive the new second derivative block hybrid method for the solution of stiff systems of ordinary differential equations of the form (1.1).

Let the power series polynomial of the form

$$y(x) = \sum_{j=0}^{r+s-1} \Omega \chi^j \tag{2.1}$$

be the approximate solution to equation (1.1) for  $x \in [x_n, x_{n+1}]$  where  $r + s$  is the sum of the number of interpolation and collocation points,  $n = 0, 1, 2, \dots, N - 1$ ,  $a's$  are the real coefficients to be determined,  $h = x_n - x_{n-1}$  is a constant step size of the partition of interval  $[a, b]$ , which is given, by the partition  $\pi_N: a = x_0 < x_1 < x_2 < \dots < x_n < x_{n+1} < \dots < x_N = b$ .

Differentiating equation (2.1) once and twice yield

$$y'(x) = \sum_{j=0}^{r+s-1} j \Omega \chi^{j-1} \tag{2.2}$$

$$y''(x) = \sum_{j=0}^{r+s-1} j(j-1) \Omega \chi^{j-2} \tag{2.3}$$

To derive derivation of second derivative block hybrid method with three off-grid points  $\left(\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right)$ , we

interpolate equation (2.1) at  $x_{n+r}, r = 0$  and collocate equations (2.2) and (2.3) at  $x_{n+s}, s = 0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, 1$ ,

which leads to the following system of equation in a matrix form as

$$\begin{bmatrix}
 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12} \\
 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & 10x_n^9 & 11x_n^{10} & 12x_n^{11} \\
 0 & 1 & 2x_{n+\frac{1}{4}} & 3x_{n+\frac{1}{4}}^2 & 4x_{n+\frac{1}{4}}^3 & 5x_{n+\frac{1}{4}}^4 & 6x_{n+\frac{1}{4}}^5 & 7x_{n+\frac{1}{4}}^6 & 8x_{n+\frac{1}{4}}^7 & 9x_{n+\frac{1}{4}}^8 & 10x_{n+\frac{1}{4}}^9 & 11x_{n+\frac{1}{4}}^{10} & 12x_{n+\frac{1}{4}}^{11} \\
 0 & 1 & 2x_{n+\frac{1}{3}} & 3x_{n+\frac{1}{3}}^2 & 4x_{n+\frac{1}{3}}^3 & 5x_{n+\frac{1}{3}}^4 & 6x_{n+\frac{1}{3}}^5 & 7x_{n+\frac{1}{3}}^6 & 8x_{n+\frac{1}{3}}^7 & 9x_{n+\frac{1}{3}}^8 & 10x_{n+\frac{1}{3}}^9 & 11x_{n+\frac{1}{3}}^{10} & 12x_{n+\frac{1}{3}}^{11} \\
 0 & 1 & 2x_{n+\frac{2}{3}} & 3x_{n+\frac{2}{3}}^2 & 4x_{n+\frac{2}{3}}^3 & 5x_{n+\frac{2}{3}}^4 & 6x_{n+\frac{2}{3}}^5 & 7x_{n+\frac{2}{3}}^6 & 8x_{n+\frac{2}{3}}^7 & 9x_{n+\frac{2}{3}}^8 & 10x_{n+\frac{2}{3}}^9 & 11x_{n+\frac{2}{3}}^{10} & 12x_{n+\frac{2}{3}}^{11} \\
 0 & 1 & 2x_{n+\frac{3}{4}} & 3x_{n+\frac{3}{4}}^2 & 4x_{n+\frac{3}{4}}^3 & 5x_{n+\frac{3}{4}}^4 & 6x_{n+\frac{3}{4}}^5 & 7x_{n+\frac{3}{4}}^6 & 8x_{n+\frac{3}{4}}^7 & 9x_{n+\frac{3}{4}}^8 & 10x_{n+\frac{3}{4}}^9 & 11x_{n+\frac{3}{4}}^{10} & 12x_{n+\frac{3}{4}}^{11} \\
 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 & 10x_{n+1}^9 & 11x_{n+1}^{10} & 12x_{n+1}^{11} \\
 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 & 72x_n^7 & 90x_n^8 & 110x_n^9 & 132x_n^{10} \\
 0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x_{n+\frac{1}{4}}^2 & 20x_{n+\frac{1}{4}}^3 & 30x_{n+\frac{1}{4}}^4 & 42x_{n+\frac{1}{4}}^5 & 56x_{n+\frac{1}{4}}^6 & 72x_{n+\frac{1}{4}}^7 & 90x_{n+\frac{1}{4}}^8 & 110x_{n+\frac{1}{4}}^9 & 132x_{n+\frac{1}{4}}^{10} \\
 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x_{n+\frac{1}{3}}^2 & 20x_{n+\frac{1}{3}}^3 & 30x_{n+\frac{1}{3}}^4 & 42x_{n+\frac{1}{3}}^5 & 56x_{n+\frac{1}{3}}^6 & 72x_{n+\frac{1}{3}}^7 & 90x_{n+\frac{1}{3}}^8 & 110x_{n+\frac{1}{3}}^9 & 132x_{n+\frac{1}{3}}^{10} \\
 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x_{n+\frac{2}{3}}^2 & 20x_{n+\frac{2}{3}}^3 & 30x_{n+\frac{2}{3}}^4 & 42x_{n+\frac{2}{3}}^5 & 56x_{n+\frac{2}{3}}^6 & 72x_{n+\frac{2}{3}}^7 & 90x_{n+\frac{2}{3}}^8 & 110x_{n+\frac{2}{3}}^9 & 132x_{n+\frac{2}{3}}^{10} \\
 0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x_{n+\frac{3}{4}}^2 & 20x_{n+\frac{3}{4}}^3 & 30x_{n+\frac{3}{4}}^4 & 42x_{n+\frac{3}{4}}^5 & 56x_{n+\frac{3}{4}}^6 & 72x_{n+\frac{3}{4}}^7 & 90x_{n+\frac{3}{4}}^8 & 110x_{n+\frac{3}{4}}^9 & 132x_{n+\frac{3}{4}}^{10} \\
 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6 & 72x_{n+1}^7 & 90x_{n+1}^8 & 110x_{n+1}^9 & 132x_{n+1}^{10}
 \end{bmatrix}
 \begin{bmatrix}
 a_0 \\
 a_1 \\
 a_2 \\
 a_3 \\
 a_4 \\
 a_5 \\
 a_6 \\
 a_7 \\
 a_8 \\
 a_9 \\
 a_{10} \\
 a_{11} \\
 a_{12}
 \end{bmatrix}
 =
 \begin{bmatrix}
 y_n \\
 f_n \\
 f_{n+\frac{1}{4}} \\
 f_{n+\frac{1}{3}} \\
 f_{n+\frac{2}{3}} \\
 f_{n+\frac{3}{4}} \\
 f_{n+1} \\
 g_n \\
 g_{n+\frac{1}{4}} \\
 g_{n+\frac{1}{3}} \\
 g_{n+\frac{2}{3}} \\
 g_{n+\frac{3}{4}} \\
 g_{n+1}
 \end{bmatrix}
 \tag{2.4}$$

Using Gaussian elimination method, (2.4) is solved for the  $a_{j,s}$ . The values of the  $a_{j,s}$  obtained are then substituted into (2.1) to give, after some simplification, a continuous block hybrid method in the form of the continuous linear multistep method as

$$y(x) = \alpha_0(x)y_n + h \left[ \beta_0(x)f_n + \beta_1(x)f_{n+\frac{1}{4}} + \beta_1(x)f_{n+\frac{1}{3}} + \beta_2(x)f_{n+\frac{2}{3}} + \beta_3(x)f_{n+\frac{3}{4}} + \beta_1(x)f_{n+1} \right] + h^2 \left[ \delta_0(x)g_n + \delta_1(x)g_{n+\frac{1}{4}} + \delta_1(x)g_{n+\frac{1}{3}} + \delta_2(x)g_{n+\frac{2}{3}} + \delta_3(x)g_{n+\frac{3}{4}} + \delta_1(x)g_{n+1} \right] \tag{2.5}$$

where  $\alpha_0(x), \beta_0(x), \beta_1(x), \beta_1(x), \beta_2(x), \beta_3(x), \beta_1(x), \delta_0(x), \delta_1(x), \delta_1(x), \delta_2(x), \delta_3(x), \delta_1(x)$  are coefficients of the continuous

scheme in equation (2.6) as

$$\alpha_0 = y_n$$

$$\beta_0 = -x_n + x - \frac{353}{4} \frac{(-x_n + x)^3}{h^2} + \frac{359129}{432} \frac{(-x_n + x)^4}{h^3} - \frac{1091953}{270} \frac{(-x_n + x)^5}{h^4} + \frac{1333709}{108} \frac{(-x_n + x)^6}{h^5} - \frac{2718239}{108} \frac{(-x_n + x)^7}{h^6} + \frac{30101945}{864} \frac{(-x_n + x)^8}{h^7} - \frac{874864}{27} \frac{(-x_n + x)^9}{h^8} + \frac{290956}{15} \frac{(-x_n + x)^{10}}{h^9} - \frac{74304}{11} \frac{(-x_n + x)^{11}}{h^{10}}$$

$$\beta_{\frac{1}{4}} = -\frac{5767168}{1125} \frac{(-x_n + x)^3}{h^2} + \frac{79364096}{1125} \frac{(-x_n + x)^4}{h^3} - \frac{1488060416}{3375} \frac{(-x_n + x)^5}{h^4} + \frac{5430738944}{3375} \frac{(-x_n + x)^6}{h^5}$$

$$- \frac{93585408}{25} \frac{(-x_n + x)^7}{h^6} + \frac{19346997248}{3375} \frac{(-x_n + x)^8}{h^7} - \frac{19467993088}{3375} \frac{(-x_n + x)^9}{h^8} + \frac{6892519424}{1875} \frac{(-x_n + x)^{10}}{h^9}$$

$$- \frac{168296448}{125} \frac{(-x_n + x)^{11}}{h^{10}} + \frac{27000832}{125} \frac{(-x_n + x)^{12}}{h^{11}}$$

$$\beta_{\frac{1}{3}} = \frac{629856}{125} \frac{(-x_n + x)^3}{h^2} - \frac{34386201}{500} \frac{(-x_n + x)^4}{h^3} + \frac{267091749}{625} \frac{(-x_n + x)^5}{h^4} - \frac{1552028607}{1000} \frac{(-x_n + x)^6}{h^5}$$

$$+ \frac{898706097}{250} \frac{(-x_n + x)^7}{h^6} - \frac{4386402477}{800} \frac{(-x_n + x)^8}{h^7} + \frac{687178728}{125} \frac{(-x_n + x)^9}{h^8} - \frac{2182529772}{625} \frac{(-x_n + x)^{10}}{h^9}$$

$$+ \frac{1753519104}{1375} \frac{(-x_n + x)^{11}}{h^{10}} - \frac{25509168}{125} \frac{(-x_n + x)^{12}}{h^{11}}$$

$$\beta_{\frac{2}{3}} = -\frac{964467}{500} \frac{(-x_n + x)^3}{h^2} + \frac{58793121}{2000} \frac{(-x_n + x)^4}{h^3} - \frac{51300459}{250} \frac{(-x_n + x)^5}{h^4} + \frac{419796837}{500} \frac{(-x_n + x)^6}{h^5}$$

$$- \frac{218894643}{100} \frac{(-x_n + x)^7}{h^6} + \frac{14969164257}{4000} \frac{(-x_n + x)^8}{h^7} - \frac{522302256}{125} \frac{(-x_n + x)^9}{h^8} + \frac{1832959692}{625} \frac{(-x_n + x)^{10}}{h^9}$$

$$- \frac{1613691072}{1375} \frac{(-x_n + x)^{11}}{h^{10}} + \frac{25509168}{125} \frac{(-x_n + x)^{12}}{h^{11}}$$

$$\beta_{\frac{3}{4}} = \frac{262144}{125} \frac{(-x_n + x)^3}{h^2} - \frac{107610112}{3375} \frac{(-x_n + x)^4}{h^3} + \frac{3745841152}{16875} \frac{(-x_n + x)^5}{h^4} - \frac{3056238592}{3375} \frac{(-x_n + x)^6}{h^5}$$

$$+ \frac{7942832128}{3375} \frac{(-x_n + x)^7}{h^6} - \frac{2706915328}{675} \frac{(-x_n + x)^8}{h^7} + \frac{15062073344}{3375} \frac{(-x_n + x)^9}{h^8} - \frac{5854429184}{1875} \frac{(-x_n + x)^{10}}{h^9}$$

$$+ \frac{155713536}{125} \frac{(-x_n + x)^{11}}{h^{10}} - \frac{27000832}{125} \frac{(-x_n + x)^{12}}{h^{11}}$$

$$\beta_1 = \frac{68}{9} \frac{(-x_n + x)^3}{h^2} - \frac{4207}{36} \frac{(-x_n + x)^4}{h^3} + \frac{112094}{135} \frac{(-x_n + x)^5}{h^4} - \frac{749495}{216} \frac{(-x_n + x)^6}{h^5} + \frac{18551}{2} \frac{(-x_n + x)^7}{h^6}$$

$$- \frac{14106425}{864} \frac{(-x_n + x)^8}{h^7} + \frac{508904}{27} \frac{(-x_n + x)^9}{h^8} - \frac{205996}{15} \frac{(-x_n + x)^{10}}{h^9} + \frac{62976}{11} \frac{(-x_n + x)^{11}}{h^{10}}$$

$$- \frac{1040}{h^{11}} \frac{(-x_n + x)^{12}}{h^{11}}$$

$$\delta_0 = \frac{1}{2} (-x_n + x)^2 - \frac{65}{9} \frac{(-x_n + x)^3}{h} + \frac{7369}{144} \frac{(-x_n + x)^4}{h^2} - \frac{9988}{45} \frac{(-x_n + x)^5}{h^3} - \frac{9988}{45} \frac{(-x_n + x)^5}{h^3}$$

$$+ \frac{22943}{36} \frac{(-x_n + x)^6}{h^4} - \frac{78793}{63} \frac{(-x_n + x)^7}{h^5} + \frac{486169}{288} \frac{(-x_n + x)^8}{h^6} - \frac{13880}{9} \frac{(-x_n + x)^9}{h^7}$$

$$+ \frac{4556}{5} \frac{(-x_n + x)^{10}}{h^8} - \frac{4556}{5} \frac{(-x_n + x)^{10}}{h^8} - \frac{3456}{11} \frac{(-x_n + x)^{11}}{h^9} + \frac{48}{h^{10}} \frac{(-x_n + x)^{12}}{h^{10}}$$

$$\delta_{\frac{1}{4}} = -\frac{16384}{75} \frac{(-x_n + x)^3}{h} + \frac{217088}{75} \frac{(-x_n + x)^4}{h^2} - \frac{790528}{45} \frac{(-x_n + x)^5}{h^3} + \frac{14098432}{225} \frac{(-x_n + x)^6}{h^4}$$

$$- \frac{3006464}{21} \frac{(-x_n + x)^7}{h^5} + \frac{48625664}{225} \frac{(-x_n + x)^8}{h^6} - \frac{48369664}{225} \frac{(-x_n + x)^9}{h^7} + \frac{16965632}{125} \frac{(-x_n + x)^{10}}{h^8}$$

$$- \frac{13565952}{275} \frac{(-x_n + x)^{11}}{h^9} - \frac{13565952}{275} \frac{(-x_n + x)^{11}}{h^9} + \frac{196608}{25} \frac{(-x_n + x)^{12}}{h^{10}}$$

$$\delta_{\frac{1}{3}} = -\frac{6561}{25} \frac{(-x_n + x)^3}{h} + \frac{91854}{25} \frac{(-x_n + x)^4}{h^2} - \frac{11707011}{500} \frac{(-x_n + x)^5}{h^3} + \frac{17417997}{200} \frac{(-x_n + x)^6}{h^4}$$

$$- \frac{144296073}{700} \frac{(-x_n + x)^7}{h^5} + \frac{51278589}{160} \frac{(-x_n + x)^8}{h^6} - \frac{8168688}{25} \frac{(-x_n + x)^9}{h^7} + \frac{26322732}{125} \frac{(-x_n + x)^{10}}{h^8}$$

$$- \frac{21415104}{275} \frac{(-x_n + x)^{11}}{h^9} + \frac{314928}{25} \frac{(-x_n + x)^{12}}{h^{10}}$$

$$\delta_{\frac{2}{3}} = -\frac{6561}{50} \frac{(-x_n + x)^3}{h} + \frac{793881}{400} \frac{(-x_n + x)^4}{h^2} - \frac{343359}{25} \frac{(-x_n + x)^5}{h^3} + \frac{5564457}{100} \frac{(-x_n + x)^6}{h^4}$$

$$- \frac{10044891}{70} \frac{(-x_n + x)^7}{h^5} + \frac{193967217}{800} \frac{(-x_n + x)^8}{h^6} - \frac{6685416}{25} \frac{(-x_n + x)^9}{h^7} + \frac{23173452}{125} \frac{(-x_n + x)^{10}}{h^8}$$

$$- \frac{20155392}{275} \frac{(-x_n + x)^{11}}{h^9} + \frac{314928}{25} \frac{(-x_n + x)^{12}}{h^{10}}$$

$$\delta_{\frac{3}{4}} = -\frac{16384}{225} \frac{(-x_n + x)^3}{h} + \frac{249856}{225} \frac{(-x_n + x)^4}{h^2} - \frac{8728576}{1125} \frac{(-x_n + x)^5}{h^3} + \frac{794624}{25} \frac{(-x_n + x)^6}{h^4}$$

$$- \frac{130736128}{1575} \frac{(-x_n + x)^7}{h^5} + \frac{1280000}{9} \frac{(-x_n + x)^8}{h^6} - \frac{35827712}{225} \frac{(-x_n + x)^9}{h^7} + \frac{14016512}{125} \frac{(-x_n + x)^{10}}{h^8}$$

$$- \frac{12386304}{275} \frac{(-x_n + x)^{11}}{h^9} + \frac{196608}{25} \frac{(-x_n + x)^{12}}{h^{10}}$$

$$\delta_1 = -\frac{1}{3} \frac{(-x_n + x)^3}{h} + \frac{31}{6} \frac{(-x_n + x)^4}{h^2} - \frac{1325}{36} \frac{(-x_n + x)^5}{h^3} + \frac{3703}{24} \frac{(-x_n + x)^6}{h^4} - \frac{34777}{84} \frac{(-x_n + x)^7}{h^5}$$

$$+ \frac{210841}{288} \frac{(-x_n + x)^8}{h^6} - \frac{7648}{9} \frac{(-x_n + x)^9}{h^7} + \frac{3116}{5} \frac{(-x_n + x)^{10}}{h^8} - \frac{2880}{11} \frac{(-x_n + x)^{11}}{h^9} + \frac{48}{h^{10}} \frac{(-x_n + x)^{12}}{h^{10}}$$

Evaluate equation (3.11) at  $\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, 1$  to obtain the new schemes of the form

$$y_{n+\frac{1}{4}} = y_n + \frac{226258721}{3114270720} h f_n - \frac{2236441}{1080000} h f_{n+\frac{1}{4}} + \frac{31528409463}{14417920000} h f_{n+\frac{1}{3}} - \frac{9123638391}{14417920000} h f_{n+\frac{2}{3}} +$$

$$\begin{aligned}
 & \frac{747113}{1080000} h f_{n+\frac{3}{4}} + \frac{7480799}{3114270720} h f_{n+1} + \frac{10716743}{7266631680} h^2 g_n - \frac{789653}{7392000} h^2 g_{n+\frac{1}{4}} - \frac{2144335761}{20185088000} h^2 g_{n+\frac{1}{3}} \\
 & - \frac{883815057}{20185088000} h^2 g_{n+\frac{2}{3}} - \frac{176389}{7392000} h^2 g_{n+\frac{3}{4}} - \frac{766649}{7266631680} h^2 g_{n+1} \\
 y_{n+\frac{1}{3}} = & y_n + \frac{67962143}{935336160} h f_n - \frac{224935936}{110716875} h f_{n+\frac{1}{4}} + \frac{2650211}{1188000} h f_{n+\frac{1}{3}} - \frac{3764273}{5940000} h f_{n+\frac{2}{3}} \\
 & + \frac{46022656}{66430125} h f_{n+\frac{3}{4}} + \frac{749909}{311778720} h f_{n+1} + \frac{643877}{436490208} h^2 g_n - \frac{60430336}{568346625} h^2 g_{n+\frac{1}{4}} - \frac{106739}{997920} h^2 g_{n+\frac{1}{3}} \\
 & - \frac{1094053}{24948000} h^2 g_{n+\frac{2}{3}} - \frac{8148992}{341007975} h^2 g_{n+\frac{3}{4}} - \frac{8539}{80831520} h^2 g_{n+1} \\
 y_{n+\frac{2}{3}} = & y_n + \frac{715703}{9743085} h f_n - \frac{552599552}{332150625} h f_{n+\frac{1}{4}} + \frac{376588}{185625} h f_{n+\frac{1}{3}} - \frac{155141}{185625} h f_{n+\frac{2}{3}} \\
 & + \frac{117440512}{110716875} h f_{n+\frac{3}{4}} + \frac{93596}{29229255} h f_{n+1} - \frac{3811}{2525985} h^2 g_n - \frac{161677312}{1705039875} h^2 g_{n+\frac{1}{4}} - \frac{59408}{779625} h^2 g_{n+\frac{1}{3}} \\
 & - \frac{58171}{779625} h^2 g_{n+\frac{2}{3}} - \frac{20119552}{568346625} h^2 g_{n+\frac{3}{4}} - \frac{9496}{68201595} h^2 g_{n+1} \\
 y_{n+\frac{3}{4}} = & y_n + \frac{8473203}{115343360} h f_n - \frac{66507}{40000} h f_{n+\frac{1}{4}} + \frac{5847445323}{2883584000} h f_{n+\frac{1}{3}} - \frac{11414821239}{14417920000} h f_{n+\frac{2}{3}} \\
 & + \frac{8799}{8000} h f_{n+\frac{3}{4}} + \frac{370317}{115343360} h f_{n+1} + \frac{243651}{161480704} h^2 g_n - \frac{233559}{2464000} h^2 g_{n+\frac{1}{4}} - \frac{61470009}{807403520} h^2 g_{n+\frac{1}{3}} \\
 & - \frac{1491400593}{20185088000} h^2 g_{n+\frac{2}{3}} - \frac{17691}{492800} h^2 g_{n+\frac{3}{4}} - \frac{112689}{807403520} h^2 g_{n+1} \\
 y_{n+1} = & y_n + \frac{721}{9504} h f_n - \frac{16384}{16875} h f_{n+\frac{1}{4}} + \frac{306909}{220000} h f_{n+\frac{1}{3}} + \frac{306909}{220000} h f_{n+\frac{2}{3}} - \frac{16384}{16875} h f_{n+\frac{3}{4}} \\
 & + \frac{721}{9504} h f_{n+1} + \frac{179}{110880} h^2 g_n - \frac{2048}{28875} h^2 g_{n+\frac{1}{4}} - \frac{9963}{308000} h^2 g_{n+\frac{1}{3}} + \frac{9963}{308000} h^2 g_{n+\frac{2}{3}} \\
 & + \frac{2048}{28875} h^2 g_{n+\frac{3}{4}} - \frac{179}{110880} h^2 g_{n+1}
 \end{aligned}
 \tag{2.6}$$

### III. ANALYSIS OF THE METHOD

The analysis of the basic properties of the new second derivative block hybrid method (2.6) were numerically analyzed. According to Tumba et al. (2018), the order of the new second derivative block hybrid method was obtained using the Taylor series expansion of equation (2.6) and it is found to be of uniform order twelve, with an error constant in a vector form as:

$$C_8 = [1.4713 \times 10^{-13}, 1.0491 \times 10^{-8}, 1.1505 \times 10^{-13}, 9.7087 \times 10^{-8}, 8.6027 \times 10^{-8}]^T$$

Following the method of Abbulimen & Igbokwe, (2017), the new second derivative block hybrid method (2.6) is consistent, since the order the new method is of uniform order eleven. In order to find the zero-stability of new second derivative block hybrid method (2.6), we consider the first characteristic polynomial of the method according to Abbulimen & Igbokwe, (2017), as follows

$$\psi(\xi) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \xi \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 0$$

$$\xi^4(\xi - 1) = 0$$

$$\Rightarrow \xi = 0, 0, 0, 0, 1$$

Hence the method is zero-stable since.

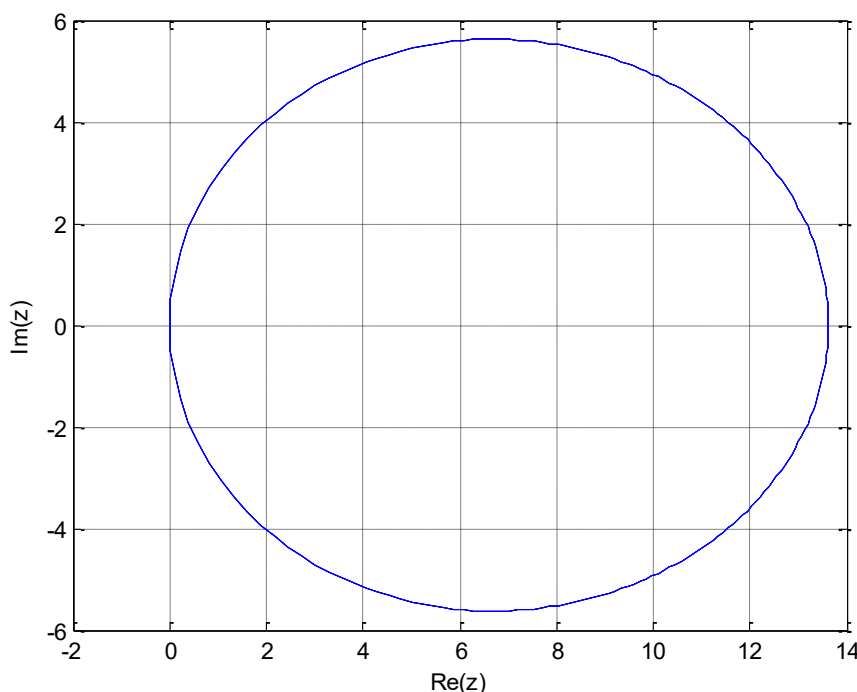
As state in Onumanyi et al. (2014), the new second derivative block hybrid method (2.6) is said to be convergent if it is consistent and zero stable. It is evident that, the new second derivative block hybrid method (2.6) is convergent.

The region of absolute stability were obtained Kuboye & Omar, (2015), using the stability polynomial of the form

$$\bar{h}(w) = \left(-\frac{1}{241920}w^5 - \frac{1}{241920}w^6\right)h^{12} + \left(-\frac{11}{103680}w^5 - \frac{11}{103680}w^6\right)h^{10} + \left(-\frac{19}{10368}w^5 + \frac{7}{4320}w^6\right)h^8 \quad (3.1)$$

$$+ \left(-\frac{29}{1728}w^5 - \frac{29}{1728}w^6\right)h^6 + \left(-\frac{65}{432}w^4 + \frac{101}{864}w^5\right)h^4 + \left(-\frac{1}{2}w^5 - \frac{1}{2}w^6\right)h^2 - 2w^4 + w^5$$

to obtained the absolute stability region of the new second derivative block hybrid method (2.6) as



**Figure 1:** Region of Absolute Stability

The stability region obtained in Figure 4.1 is A-stable, since it contains the whole of the left-half plane of the figure. Note that the unstable region is the exterior of the curve while the stability region is the interior of the curve.

#### IV. NUMERICAL RESULTS

The simulation of second derivative block hybrid method for solving stiff systems of ordinary differential equation (1.1) were carryout in this section. Some systems of equations were consider to validate efficiency of the new method.

Example 4.1

Consider a two-dimensional stiff system of differential equations which was solved by Skwame et al. (2017), of the form

$$\left. \begin{aligned} y_1' &= 998y_1 + 1998y_2 & y_1(0) &= 1 \\ y_2' &= -999y_1 - 1999y_2 & y_2(0) &= 0 \end{aligned} \right\} \quad (4.1)$$

With Exact Solution

$$\left. \begin{aligned} y_1(x) &= 2e^{-x} - e^{-1000x} \\ y_2(x) &= -e^{-x} - e^{-1000x} \end{aligned} \right\} \quad (4.2)$$

$$x \in [0, 1], \quad h = 0.1$$

(See Skwame et-al 2017).

**Table 4.1:** Numerical Solutions of Problem 4.1 showing the Result of  $y_1$

| x   | Exact Solution         | Computed solution      | Error in New Method | Error in Skwame et al. (2017) |
|-----|------------------------|------------------------|---------------------|-------------------------------|
| 0.1 | 0.90483741803595957316 | 0.90483741803595964261 | 6.9450E-17          | 2.4320E-02                    |
| 0.2 | 0.81873075307798185867 | 0.81873075307798191813 | 5.9460E-17          | 3.8710E-02                    |
| 0.3 | 0.74081822068171786607 | 0.74081822068171791926 | 5.3190E-17          | 9.3060E-04                    |
| 0.4 | 0.67032004603563930074 | 0.67032004603563934869 | 4.7950E-17          | 1.5090E-03                    |
| 0.5 | 0.60653065971263342360 | 0.60653065971263346688 | 4.3280E-17          | 2.3200E-05                    |
| 0.6 | 0.54881163609402643263 | 0.54881163609402647122 | 3.8590E-17          | 6.9930E-05                    |
| 0.7 | 0.49658530379140951470 | 0.49658530379140954979 | 3.5090E-17          | 2.1470E-05                    |
| 0.8 | 0.44932896411722159143 | 0.44932896411722162290 | 3.1470E-17          | 2.3390E-05                    |
| 0.9 | 0.40656965974059911188 | 0.40656965974059914018 | 2.8300E-17          | 2.1690E-05                    |
| 1.0 | 0.36787944117144232160 | 0.36787944117144234682 | 2.5220E-17          | 1.9710E-05                    |

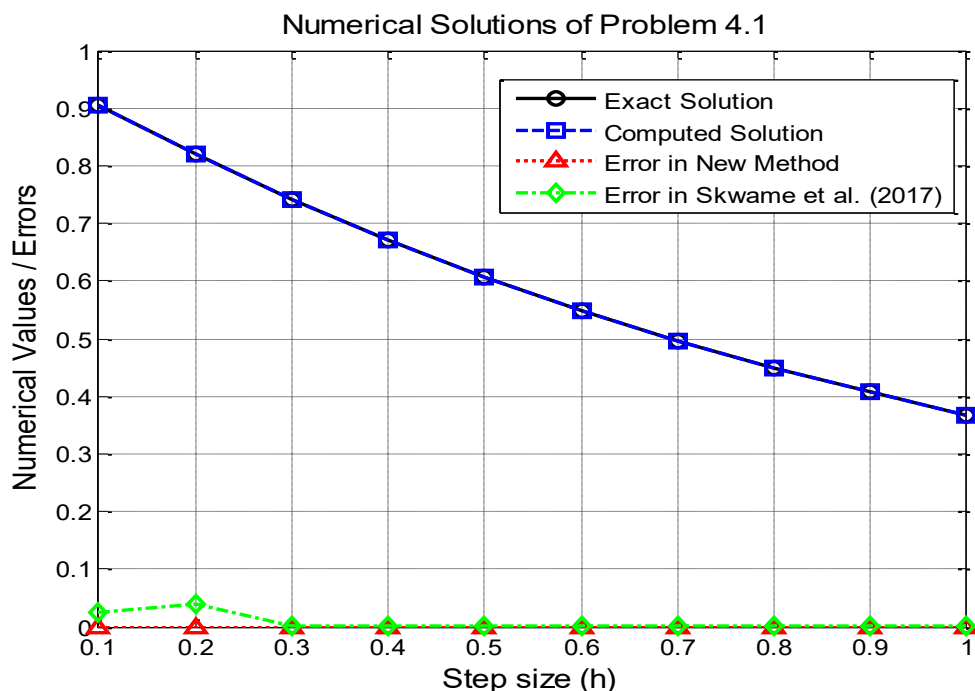


Figure 2: Graphical Curve of Table 4.1

Table 4.2: Numerical Solutions of Problem 4.1 showing the Result of  $y_2$

| x   | Exact Solution          | Computed solution       | Error in New Method | Error in Skwame et al. (2017) |
|-----|-------------------------|-------------------------|---------------------|-------------------------------|
| 0.1 | -0.90483741803595957316 | -0.90483741803595971266 | 1.3950E-15          | 2.4320E-02                    |
| 0.2 | -0.81873075307798185867 | -0.81873075307798197859 | 1.1992E-15          | 3.8110E-02                    |
| 0.3 | -0.74081822068171786607 | -0.74081822068171797380 | 1.0773E-15          | 9.3260E-04                    |
| 0.4 | -0.67032004603563930074 | -0.67032004603563939816 | 9.7420E-16          | 1.5070E-03                    |
| 0.5 | -0.60653065971263342360 | -0.60653065971263351183 | 8.8230E-16          | 2.2030E-05                    |
| 0.6 | -0.54881163609402643263 | -0.54881163609402651158 | 7.8950E-16          | 7.1350E-05                    |
| 0.7 | -0.49658530379140951470 | -0.49658530379140958678 | 7.2080E-16          | 1.2230E-05                    |
| 0.8 | -0.44932896411722159143 | -0.44932896411722165629 | 6.4860E-16          | 1.4550E-05                    |
| 0.9 | -0.40656965974059911188 | -0.40656965974059917046 | 5.8580E-16          | 1.6040E-05                    |
| 1.0 | -0.36787944117144232160 | -0.36787944117144237397 | 5.2370E-16          | 1.4830E-05                    |

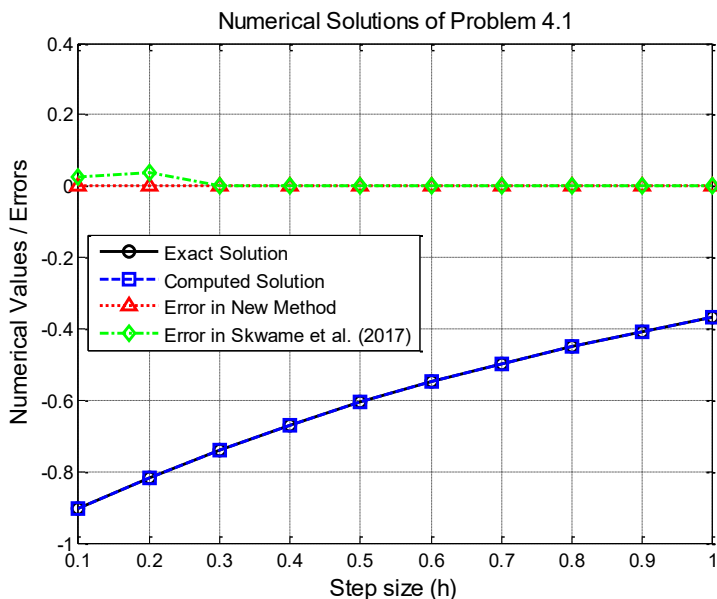


Figure 3: Graphical Curve of Table 4.2

Example 4.2: Consider a two-dimensional stiff system of differential equations which was solved by Tumba et al. (2018), of the form

$$\left. \begin{aligned} y_1' &= -8y_1 + 7y_2 & y_1(0) &= 1 \\ y_2' &= 42y_1 - 43y_2 & y_2(0) &= 8 \end{aligned} \right\} \quad (4.3)$$

With Exact Solution

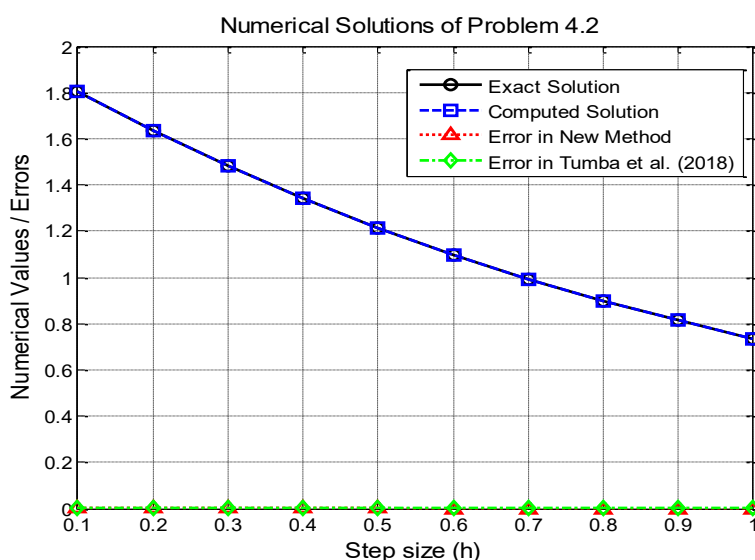
$$\left. \begin{aligned} y_1(x) &= 2e^{-x} - e^{-50x} \\ y_2(x) &= 2e^{-x} - 6e^{-50x} \end{aligned} \right\} \quad (4.4)$$

$$x \in [0, 1], \quad h = 0.1$$

(See Tumba et al. 2018)

**Table 4.3:** Numerical Solutions of Problem 4.2 showing the Result of  $y_1$

| X   | Exact Solution         | Computed solution      | Error in New Method | Error in Tumba et al. 2018 |
|-----|------------------------|------------------------|---------------------|----------------------------|
| 0.1 | 1.80293688907283367920 | 1.80293686204423168460 | 2.7028E-08          | 3.6200E-04                 |
| 0.2 | 1.63741610622620123240 | 1.63741610586196592710 | 3.6424E-10          | 3.2600E-06                 |
| 0.3 | 1.48163613546111523030 | 1.48163613545743392640 | 3.6813E-12          | 2.6000E-08                 |
| 0.4 | 1.34064009001012497910 | 1.34064009001009190760 | 3.3072E-14          | 5.0000E-09                 |
| 0.5 | 1.21306131941137890330 | 1.21306131941137862610 | 2.7720E-16          | 8.0000E-09                 |
| 0.6 | 1.09762327218795928910 | 1.09762327218795928840 | 7.0000E-19          | 8.0000E-09                 |
| 0.7 | 0.99317060758281839889 | 0.99317060758281840050 | 1.6100E-18          | 7.2000E-09                 |
| 0.8 | 0.89865792823444317861 | 0.89865792823444318028 | 1.6700E-18          | 8.2000E-09                 |
| 0.9 | 0.81313931948119822373 | 0.81313931948119822545 | 1.7200E-18          | 8.4000E-09                 |
| 1.0 | 0.73575888234288464491 | 0.73575888234288464320 | 1.7100E-18          | 8.5000E-09                 |



**Figure 4:** Graphical Curve of Table 4.3

**Table 4.4:** Numerical solutions of problem 4.1 showing the result of  $y_2$

| x   | Exact Solution         | Computed solution      | Error in New Method | Error in Tumba et al. 2018 |
|-----|------------------------|------------------------|---------------------|----------------------------|
| 0.1 | 1.76924715407740634370 | 1.85010268023804392030 | 8.0856E-02          | 8.3200E-02                 |
| 0.2 | 1.63718910657738880820 | 1.63773390791995046510 | 5.4480E-04          | 1.3200E-01                 |
| 0.3 | 1.48163827679944657440 | 1.48163460594951272110 | 3.6709E-06          | 3.9500E-06                 |
| 0.4 | 1.34064007970435686690 | 1.34064010443839877450 | 2.4734E-08          | 3.2000E-08                 |
| 0.5 | 1.21306131934193918400 | 1.21306131950859618440 | 1.6666E-10          | 7.0000E-09                 |
| 0.6 | 1.09762327218749140790 | 1.09762327218861433820 | 1.1229E-12          | 6.0000E-09                 |
| 0.7 | 0.99317060758281524633 | 0.99317060758282281503 | 7.5687E-15          | 8.4000E-09                 |
| 0.8 | 0.89865792823444315737 | 0.89865792823444321055 | 5.3180E-17          | 7.6000E-09                 |
| 0.9 | 0.81313931948119822359 | 0.81313931948119822660 | 3.0100E-18          | 8.5000E-09                 |
| 1.0 | 0.73575888234288464320 | 0.73575888234288464569 | 2.4900E-18          | 8.0000E-09                 |

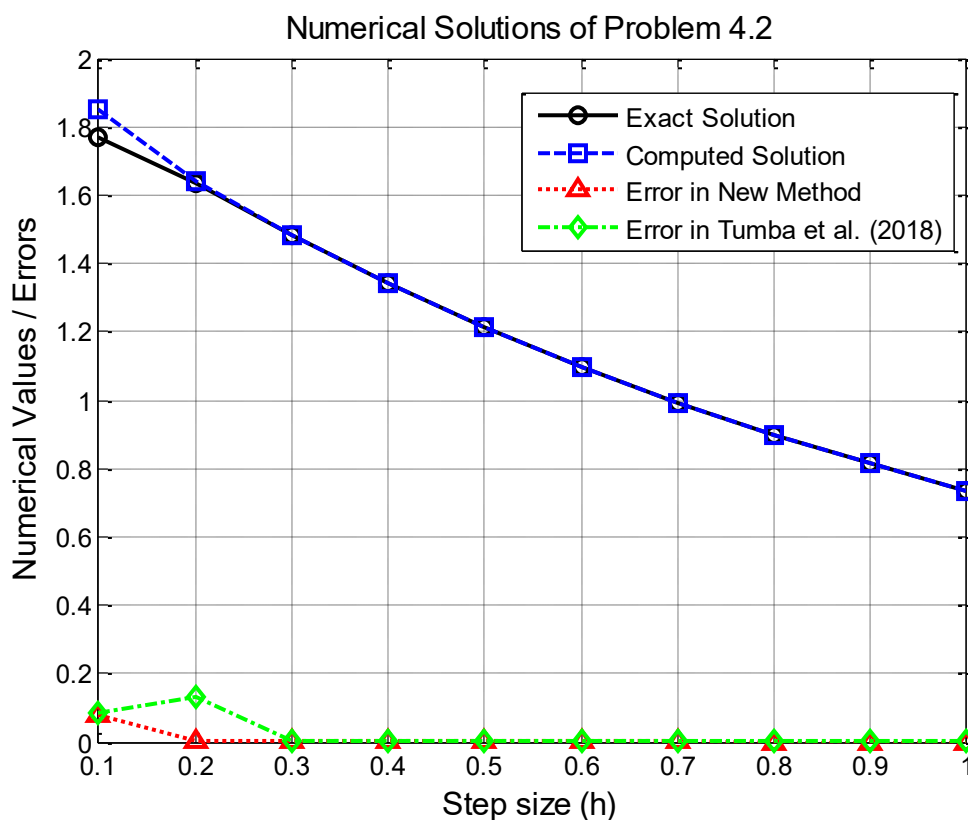


Figure 5: Graphical Curve of Table 4.4

## V. DISCUSSION OF RESULTS

The numerical results presented in Tables 4.1 and 4.2, together with Figures 2 and 3, illustrate the performance of the second derivative block hybrid method when applied to Example 4.1, a two-dimensional stiff system earlier investigated by Skwame et al. (2017). From the tables, it is evident that the computed solutions are in excellent agreement with the exact solutions over the entire interval of integration. The closeness between the two sets of solutions shows that the proposed method accurately captures the behavior of the stiff system. This observation is further supported by Figures 2 and 5, where the numerical solution curves coincide almost perfectly with the exact solution curves, indicating high precision and numerical stability. Table 4.2 specifically represents the second component of the stiff system in Example 4.1. The computed results again closely match the corresponding exact solutions at all grid points considered. The graphical illustration in Figure 3 shows a smooth and continuous overlap between the numerical and analytical solutions, with no visible deviations. This confirms the ability of the method to handle multiple components of stiff systems simultaneously while maintaining consistency and reliability in the computed results.

The effectiveness of the method is further demonstrated in Example 4.2 through the results presented in Table 4.3 and Figure 4. The numerical solutions obtained are virtually identical to the exact solutions throughout the interval considered, showing rapid convergence of the scheme. The plotted curves in Figure 4 reveal that the numerical solution faithfully tracks the exact solution at all points, highlighting the robustness of the method and its suitability for solving stiff systems encountered in practical applications. Finally, Table 4.4 and Figure 5 provide additional confirmation of the accuracy and stability of the proposed second derivative block hybrid method. Apart from the initial step, the numerical

solutions quickly align with the exact solutions and maintain this agreement across the domain. The graphical curve in Figure 5 shows that the computed solution closely follows the exact solution without oscillations or divergence. Overall, the figures and tables collectively verify that the proposed method is efficient, stable, and highly effective for solving stiff systems of ordinary differential equations. The findings from both examples confirm that the present approach delivers superior accuracy and stability compared with existing techniques. By producing more precise results with lower error margins, the method proves to be a dependable alternative for handling stiff systems effectively.

## VI. CONCLUSION

This study introduces a new second-derivative block hybrid method designed to solve stiff ordinary differential equations (ODEs) more efficiently. The scheme is constructed through a power series expansion combined with interpolation and collocation at both on-grid and off-grid points, resulting in a continuous block formulation. By incorporating second derivatives, the method enhances both stability and accuracy, making it particularly effective for stiff systems. An analysis of the method revealed several desirable properties, including accuracy, consistency, convergence, and zero-stability. In addition, it was shown to attain a uniform order of twelve while maintaining a high level of precision in practical applications. The method also possesses a wide region of absolute stability, allowing it to handle stiff problems without requiring excessively small step sizes. Numerical experiments with standard stiff test equations confirmed the method's efficiency and reliability. Compared to existing approaches, it consistently produced smaller errors and reduced computational effort. These results highlight its capability as a robust tool for addressing stiff ODEs encountered in diverse areas of science and engineering. In conclusion, the proposed second-derivative block hybrid method combines improved accuracy, stability, and computational efficiency, offering a dependable solution for stiff systems. Its potential application to complex dynamic models demonstrates its relevance and promise in real-world problem-solving.

## ACKNOWLEDGMENT

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